# A theory for large-scale structure formation in atmospheric/oceanic turbulence:

## Is jet formation a phase transition phenomenon?

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10 February 2016

# structure of talk

- introduction to the physical problem
- formulation of the theory (S3T)
- the homogeneous turbulent state and its stability
- comparison of S3T predictions with direct numerical simulations and verification of the theory
- stability of inhomogeneous turbulent states & relation with jet mergers
- summary

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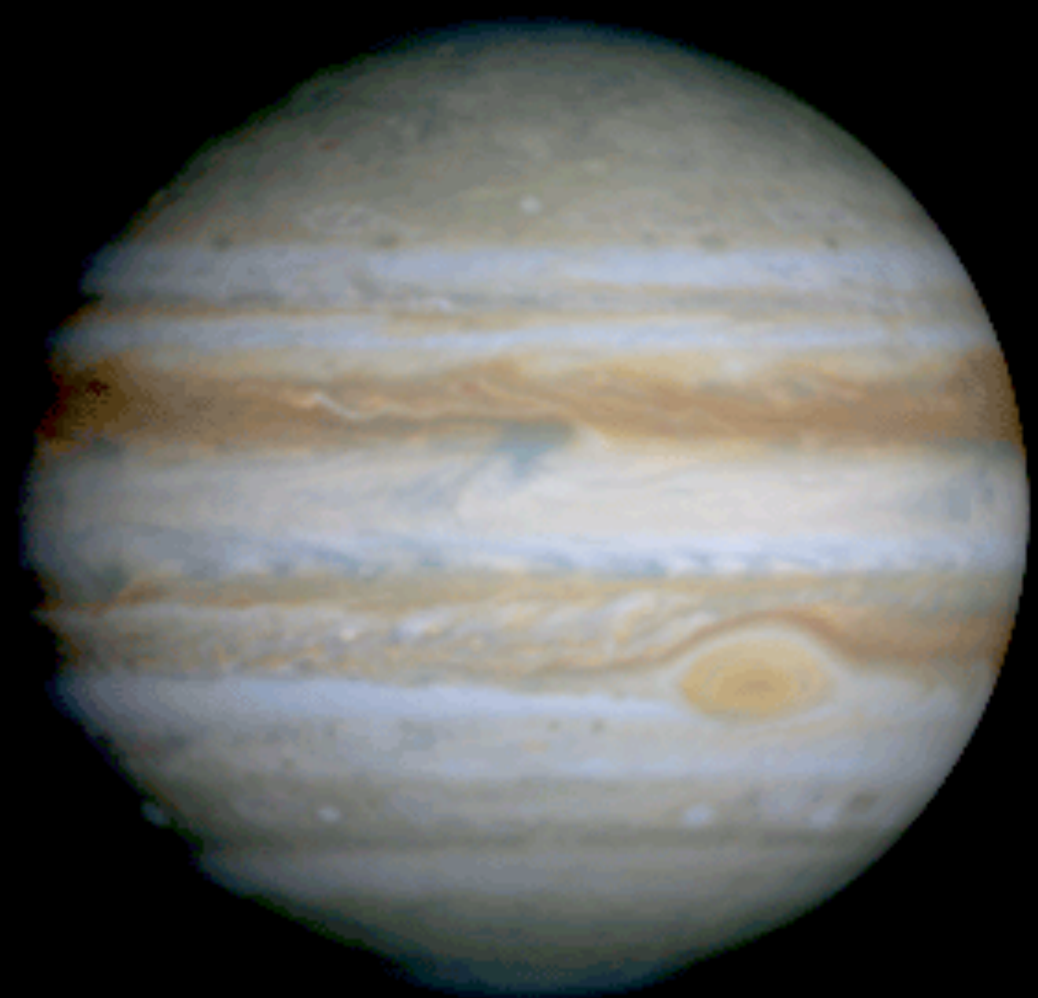
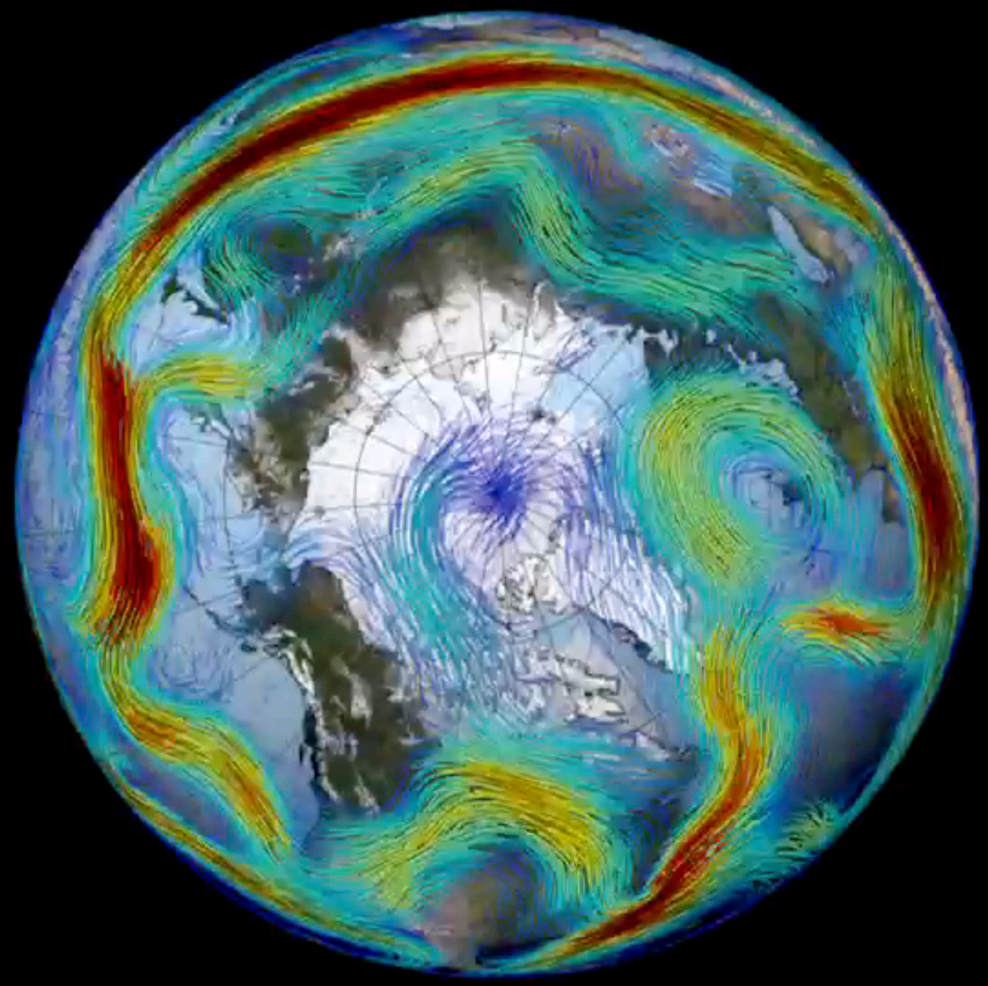
# Planetary turbulence is anisotropic and inhomogeneous



banded Jovian jets

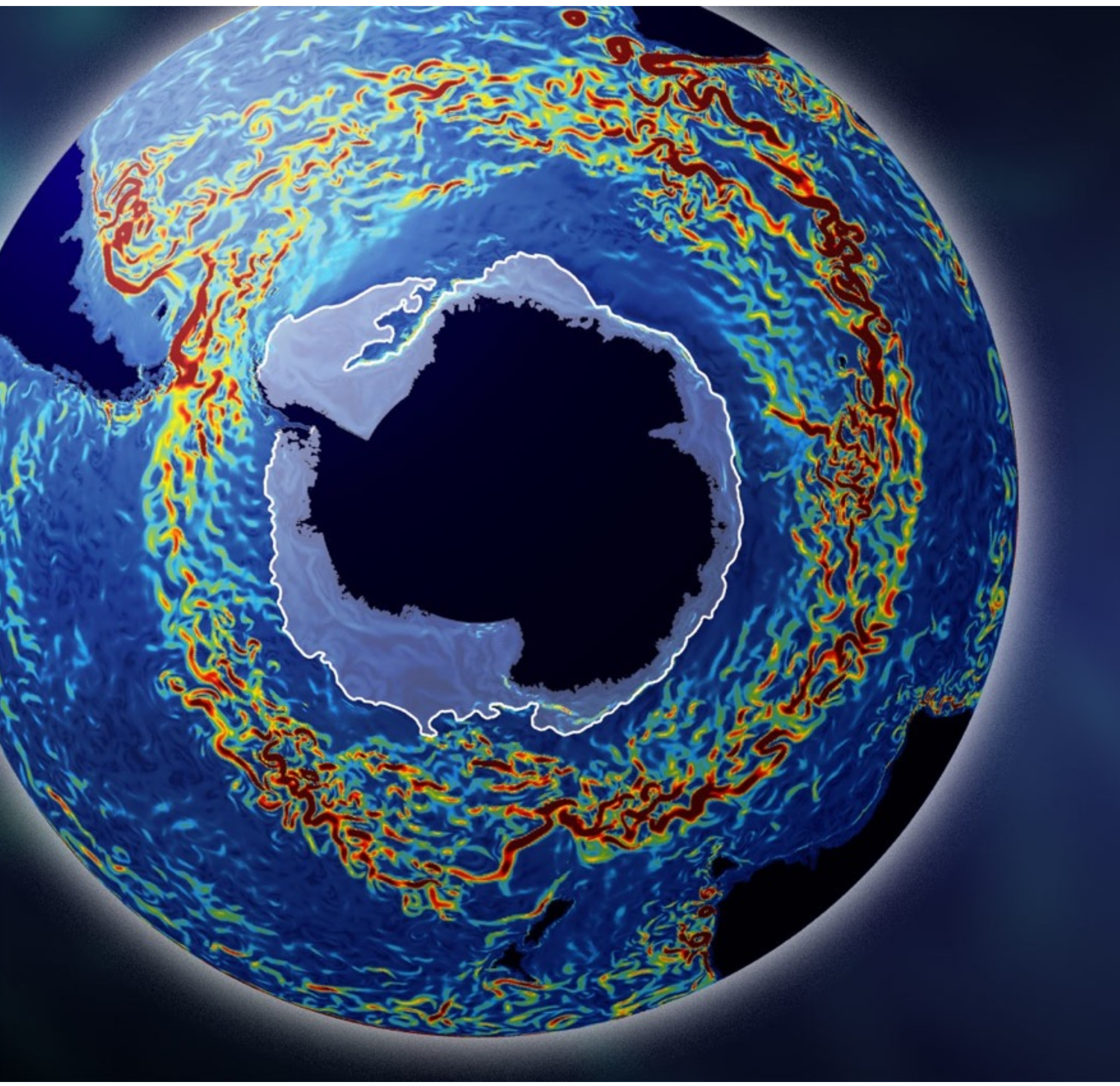
NASA/Cassini Jupiter Images

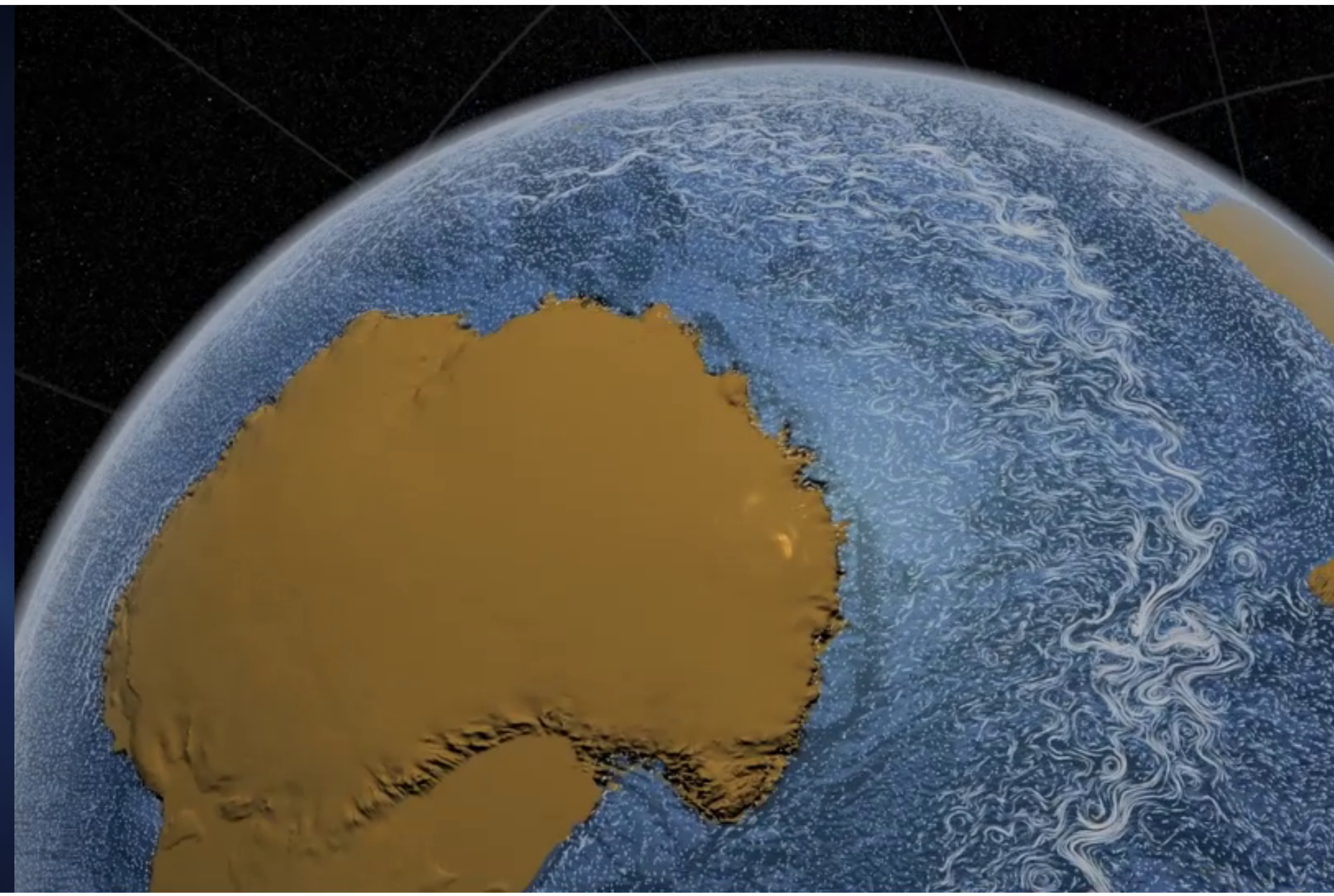


polar front jet

NASA/Goddard Space Flight Center

# Planetary turbulence is anisotropic and inhomogeneous II





computer simulation

San Diego Supercomputer Center, UCSD

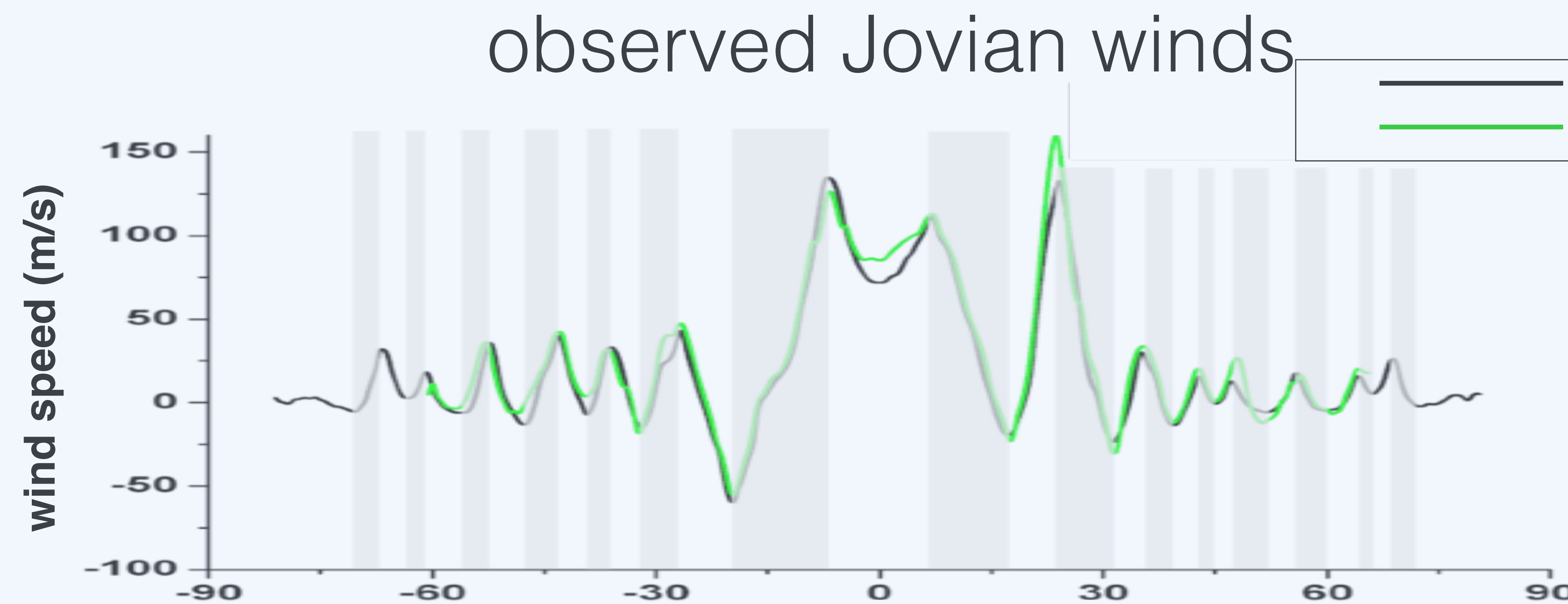
ACC

satellite observations

NASA/Goddard Space Flight Center

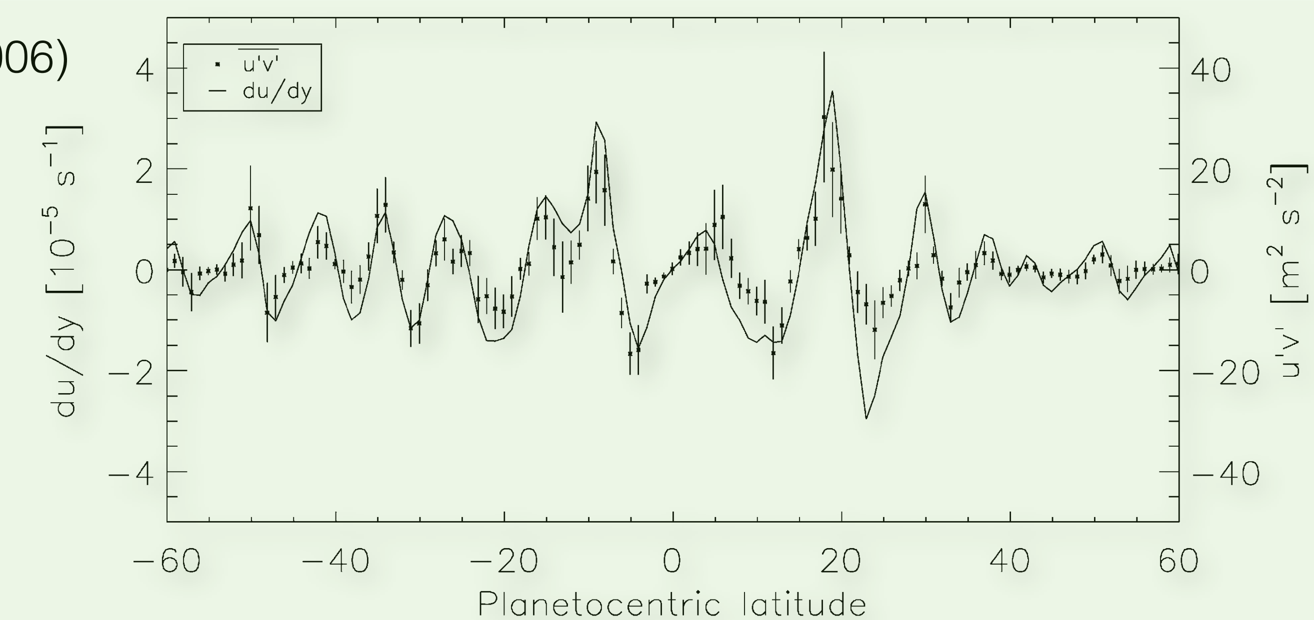
# Jets appear “steady” and are eddy-driven

“steady”




eddy-driven

(Salyk et. al. 2006)




$$\overline{u'v'} \approx \kappa \frac{\partial \overline{u}}{\partial y}$$

$$\kappa \approx 10^6\,\mathrm{m^2 s^{-1}}$$

$$\frac{\partial \overline{u}}{\partial t} = -\partial_y \overline{u'v'}$$

# Jets appear "steady" and are eddy-driven

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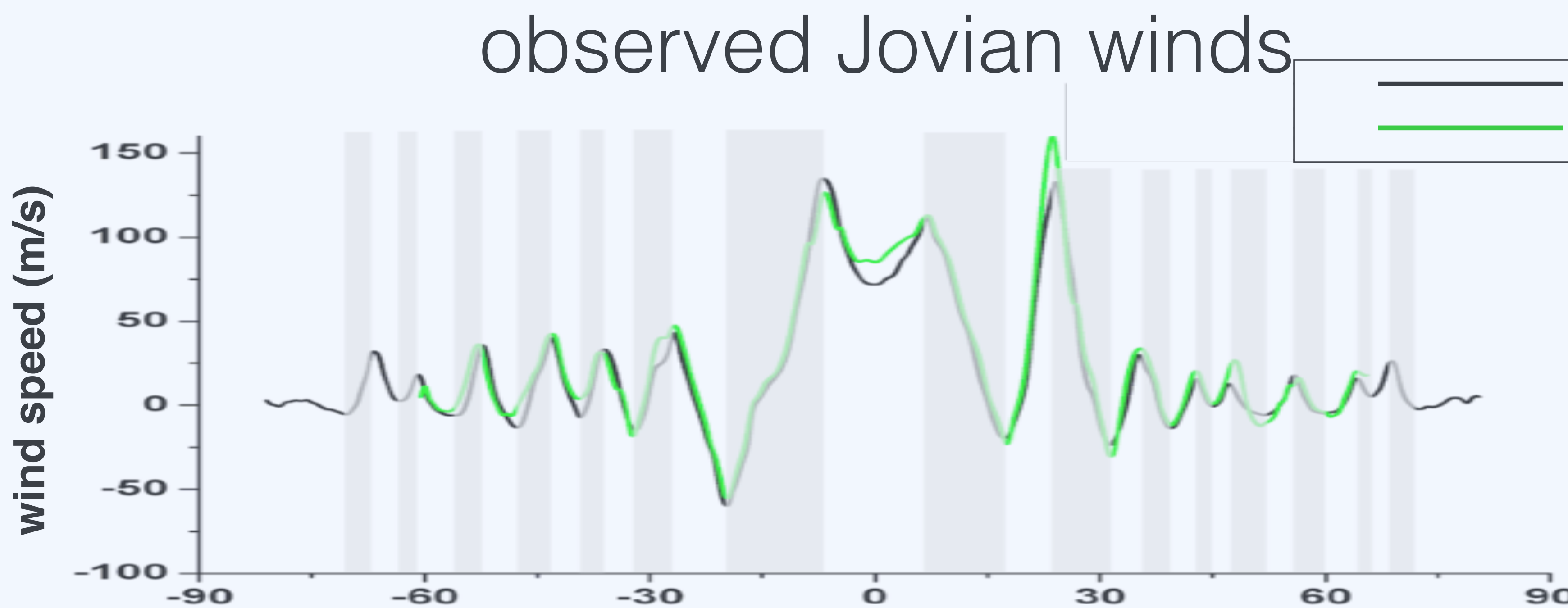
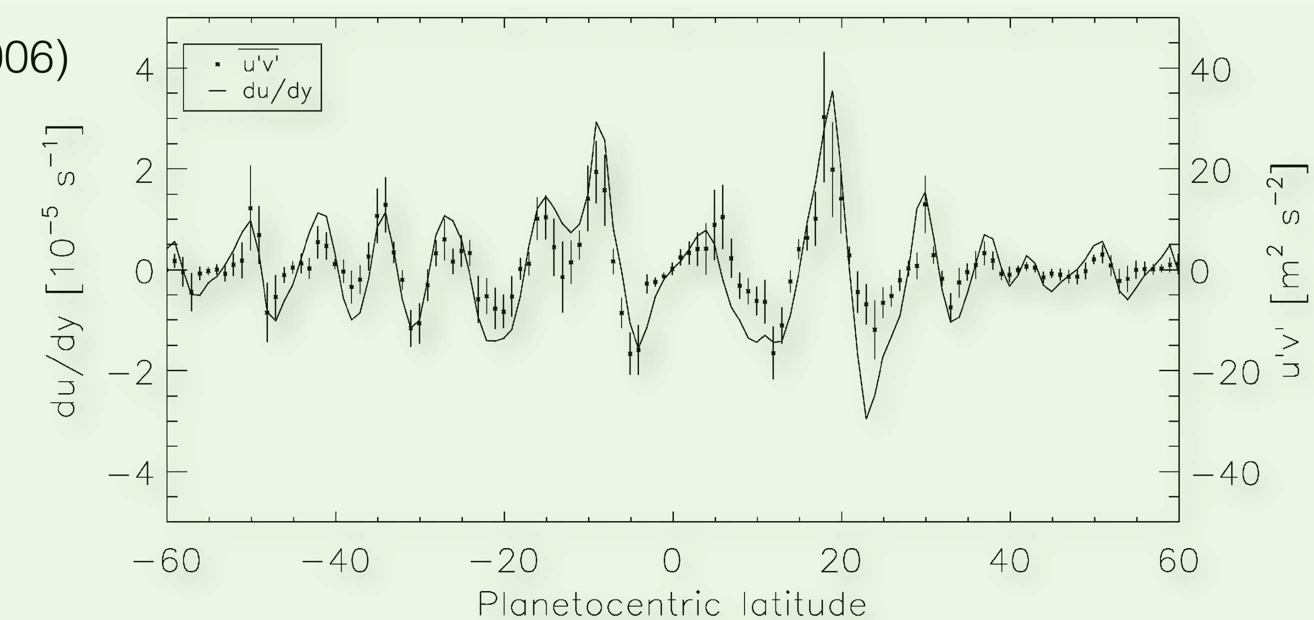



eddy-driven

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$$\frac{\partial \overline{u}}{\partial t} = -\partial_y \overline{u'v'} = -\kappa \frac{\partial^2 \overline{u}}{\partial y^2}$$

anti-diffusion

(or negative viscosity)

## $O(10)$ theoretical explanations for jet formation

most of them disagree in a large extent with each other despite the fact that everybody can produce jets numerically in simple models

(I won't attempt to survey)

# barotropic vorticity equation on a $\beta$-plane

$$\partial_t \zeta + \mathbf{u}\cdot\nabla\zeta + \beta v = -r\zeta + \sqrt{\varepsilon}\,\xi$$

linear dissipation at rate *r*

stochastic forcing

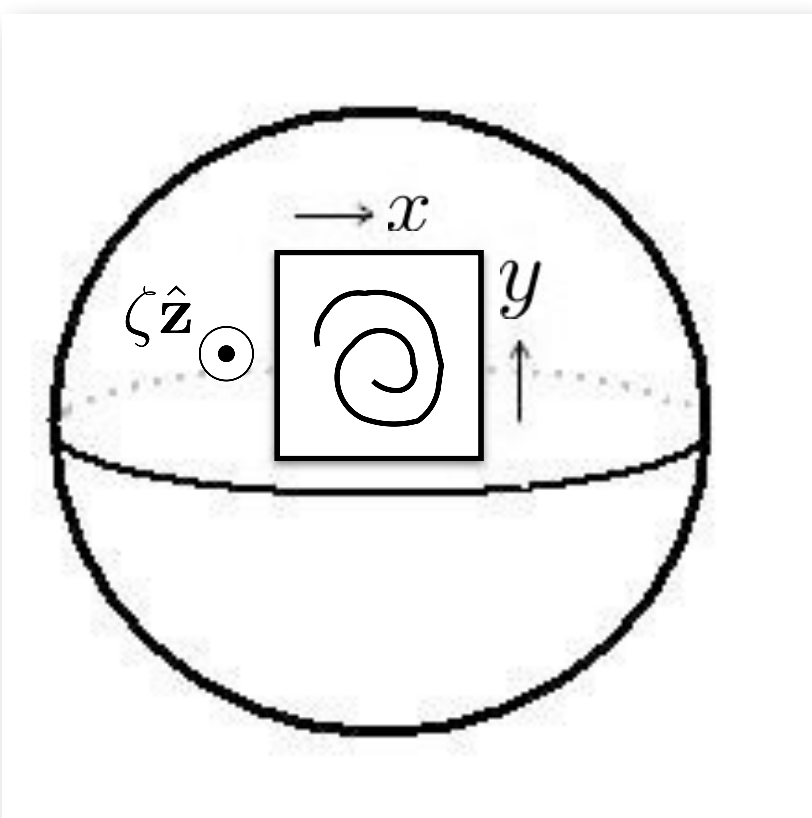
$$\nabla\cdot\mathbf{u} = 0$$

$$\mathbf{u} = (u, v) = (-\partial_y\psi, \partial_x\psi)$$

$$\zeta = (\nabla\times\mathbf{u})\cdot\hat{\mathbf{z}} = \Delta\psi$$

$$\langle \xi(\mathbf{x}_a, t)\xi(\mathbf{x}_b, t')\rangle = Q(\mathbf{x}_a - \mathbf{x}_b)\,\delta(t-t')$$

$\xi$ is statistically homogeneous




$\beta$ is the gradient of the planetary vorticity

we have two non-dimensional parameters

$\varepsilon k_f^2/r^3$

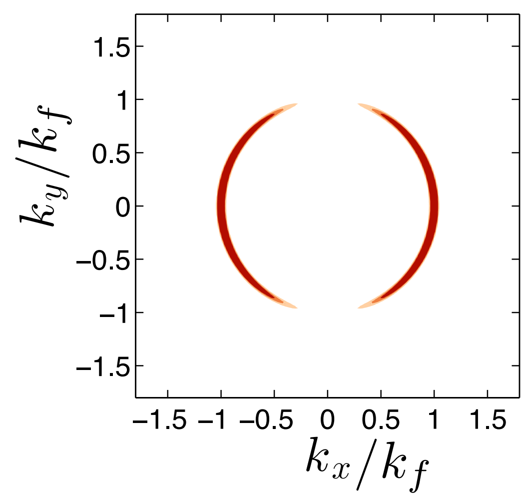
$\beta/(k_f r)$

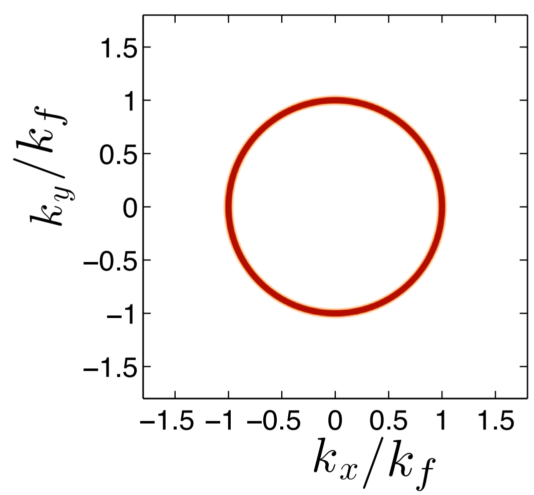
# what does the forcing look like and what does it model?

anisotropic
[≈Earth]

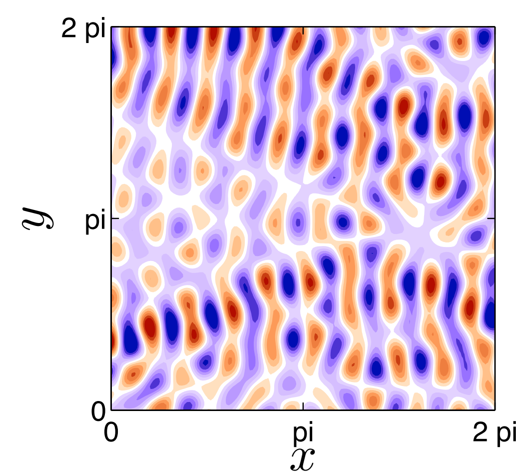
isotropic
[≈Jupiter]

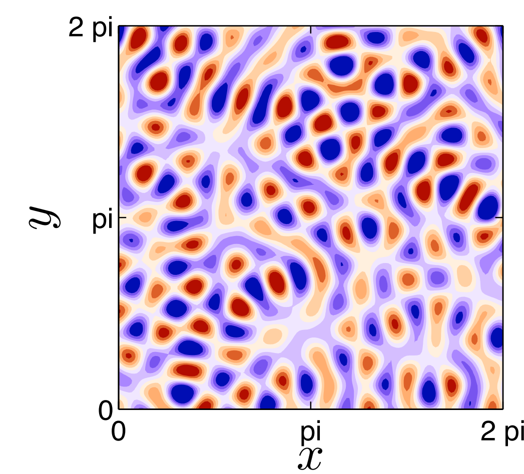
spectrum of the covariance $\hat{Q}(\mathbf{k})$




$\xi(\mathbf{x}, t)$

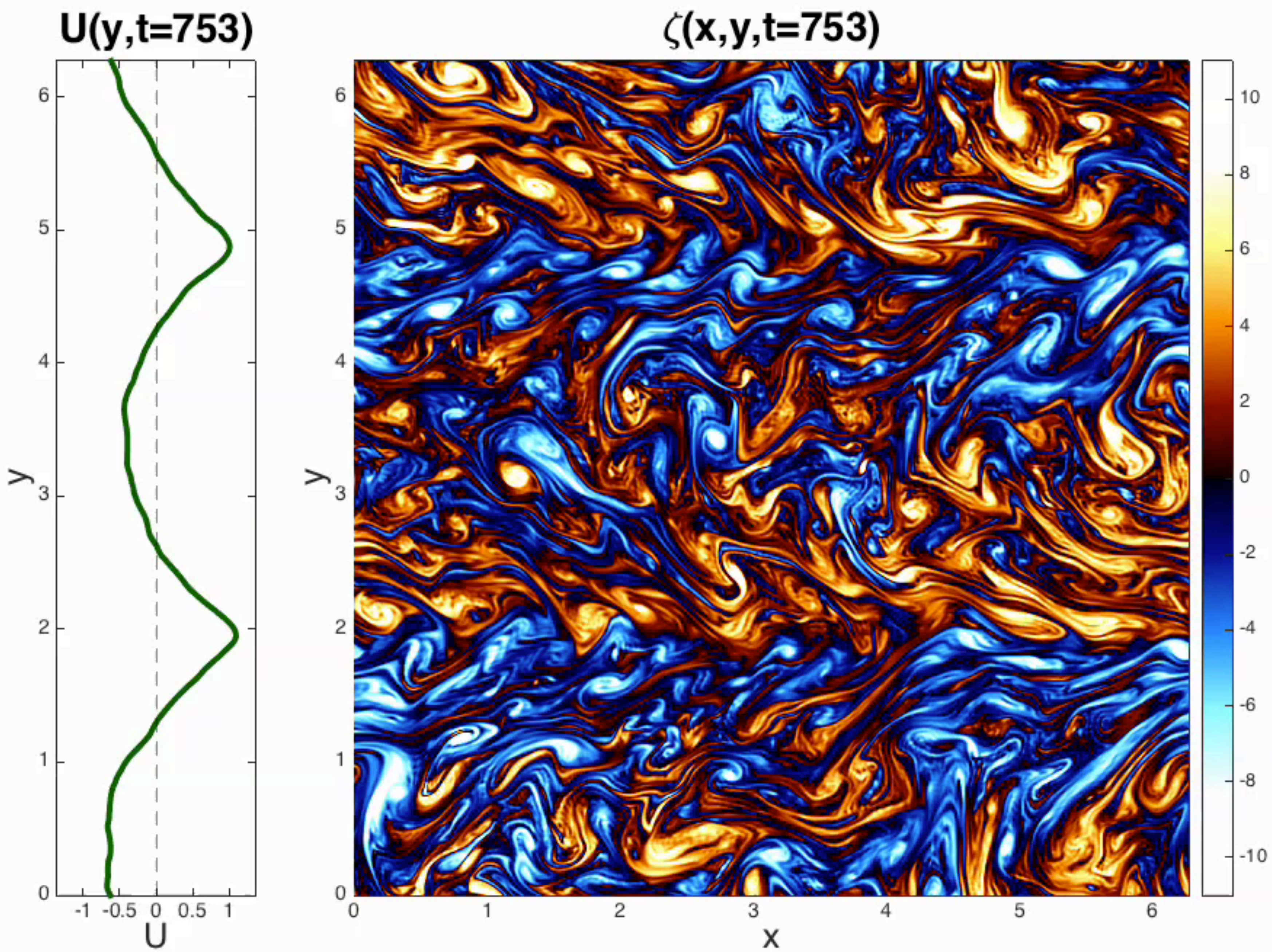



modeling energy injected
to the barotropic mode
by baroclinic instability

modeling energy injected
to the barotropic mode
by convection

# barotropic $\beta$-plane turbulence exhibits large-scale structure formation




$$\varepsilon k_f^2/r^3 = 10^6$$

$$\beta/(k_f r) = 67$$

http://www.youtube.com/watch?v=ZtzW25NooDk

statistically homogeneous forcing

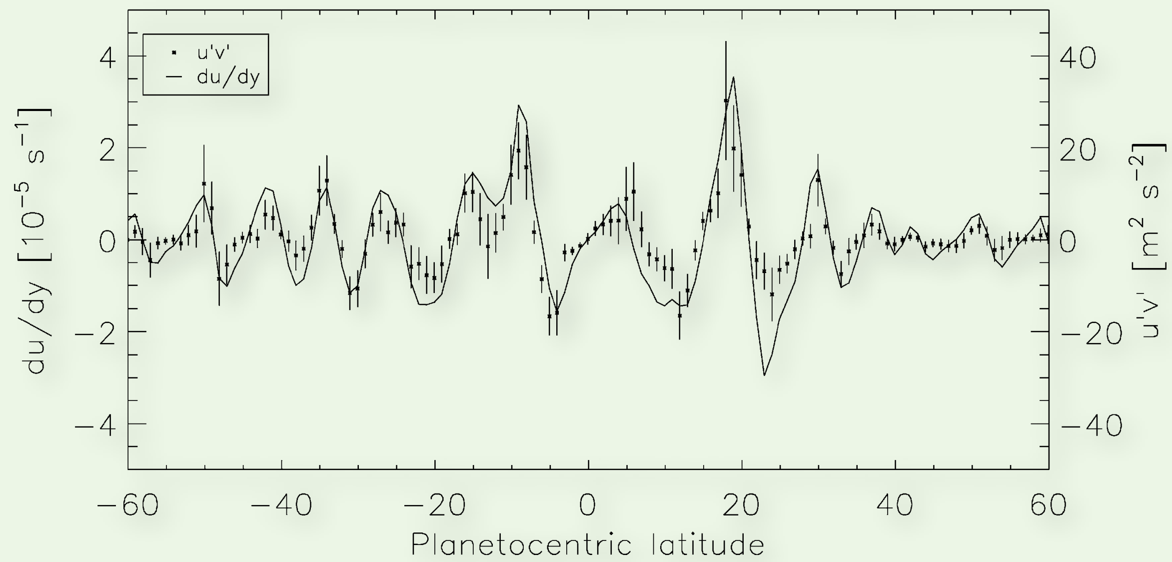
(no inhomogeneity is imposed by the forcing)

any random flow inhomogeneities organize the turbulence in a manner so that they are reinforced

we observe:

- jet emerge
- jets appear to change much slower compared to the eddies
- jet have a particular structure
- jets may merge

# remember the observations:




$$\overline{u'v'} \approx \kappa \frac{\partial \overline{u}}{\partial y}$$

$$\kappa \approx 10^6 \, \mathrm{m^2 s^{-1}}$$

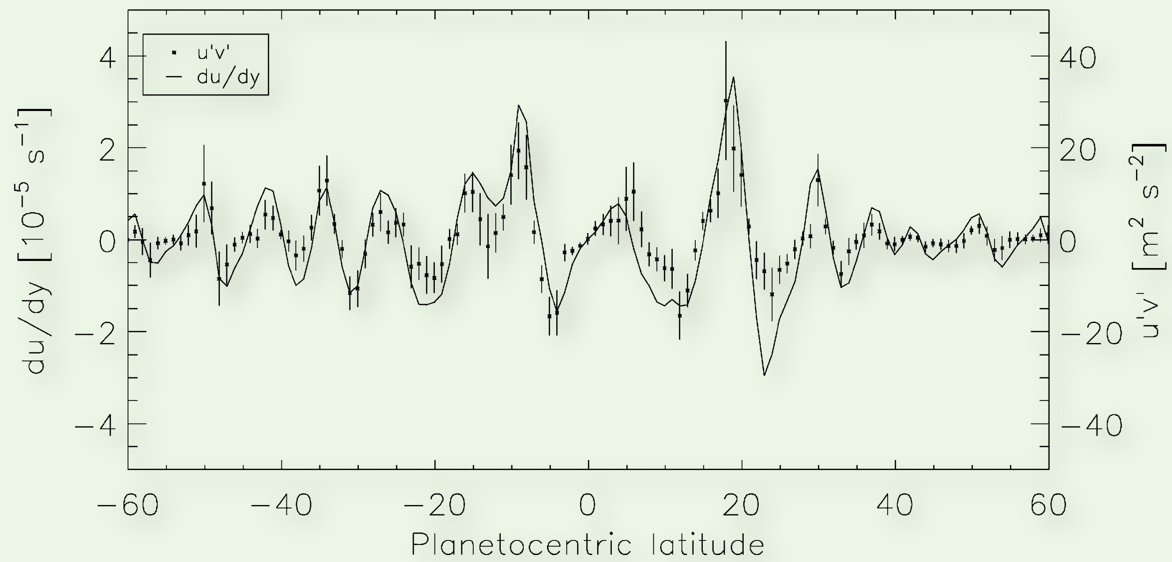
in a barotropic model zonal mean flow evolves under

$$\frac{\partial \overline{u}}{\partial t} = \overline{v'\zeta'} - r\overline{u}$$

$$= -\partial_y \overline{u'v'}$$

Reynolds stress divergence

remember the observations:

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Reynolds stress divergence

At steady state a non-zero zonal mean flow *requires* non-zero mean Reynolds stress divergence

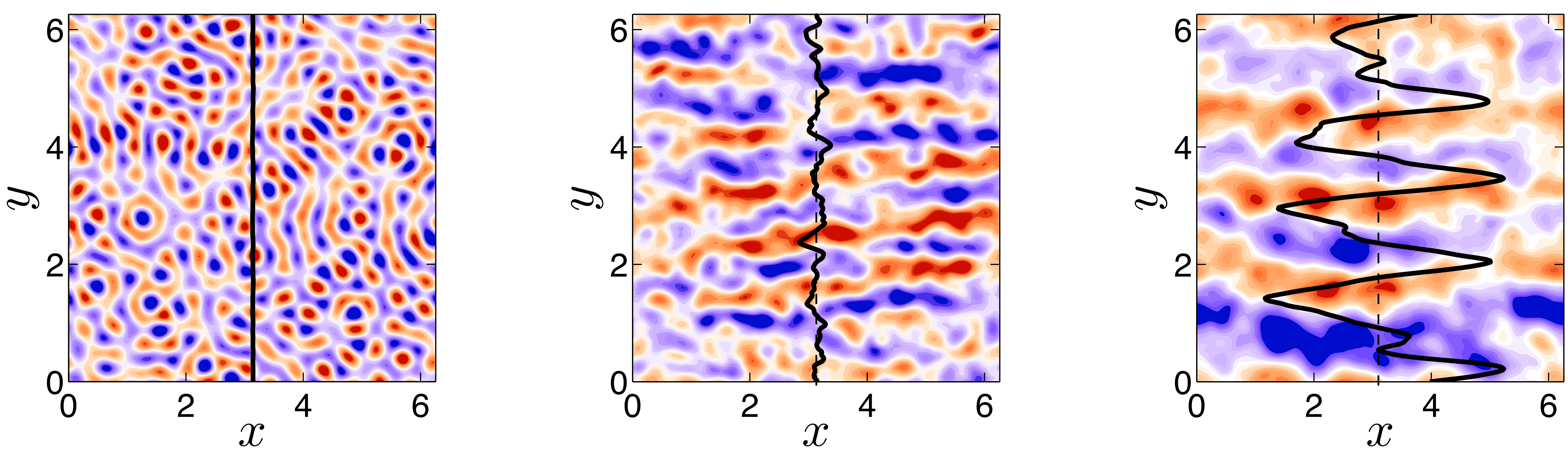
But how does a *homogeneous* stochastic excitation produce *inhomogeneous* Reynolds stress divergence?

# various $\beta$-plane turbulence flows at statistically steady state:

## homogeneous — traveling waves — zonal jets

$$\beta/(k_f r) = 67$$

$\varepsilon k_f^2/r^3 \ = 10^2$ $\qquad$ $5 \times 10^3$ $\qquad$ $5 \times 10^4$




this suggests that there is some kind of transition as $\varepsilon$ is increased

[ snapshots of the streamfunction $\psi(\mathbf{x},t)$ with instantaneous zonal mean flow $U(y,t)$ ]

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$$\partial_t \zeta + \mathbf{u} \cdot \nabla \zeta + \beta v = -r\zeta + \sqrt{\varepsilon}\, \xi$$

# barotropic vorticity equation on a $\beta$-plane

Using decomposition: $\zeta(\mathbf{x},t) = \underbrace{\langle\zeta(\mathbf{x},t)\rangle}_{Z(\mathbf{x},t)} \; + \; \zeta'(\mathbf{x},t)$

$$\partial_t Z + \mathbf{U}\cdot\nabla Z + \beta V = -\left\langle \mathbf{u}'\cdot\nabla\zeta'\right\rangle - rZ$$

$$\partial_t \zeta' = \mathcal{A}(\mathbf{U})\,\zeta' + \langle \mathbf{u}'\cdot\nabla\zeta'\rangle - \mathbf{u}'\cdot\nabla\zeta' + \sqrt{\varepsilon}\,\xi$$

with

$$\mathcal{A}(\mathbf{U}) \overset{\text{def}}{=} -\mathbf{U}\cdot\nabla + \Big[(\Delta\mathbf{U}) - \beta\partial_x\Big]\,\Delta^{-1} - r$$

$\langle\,\cdot\,\rangle$

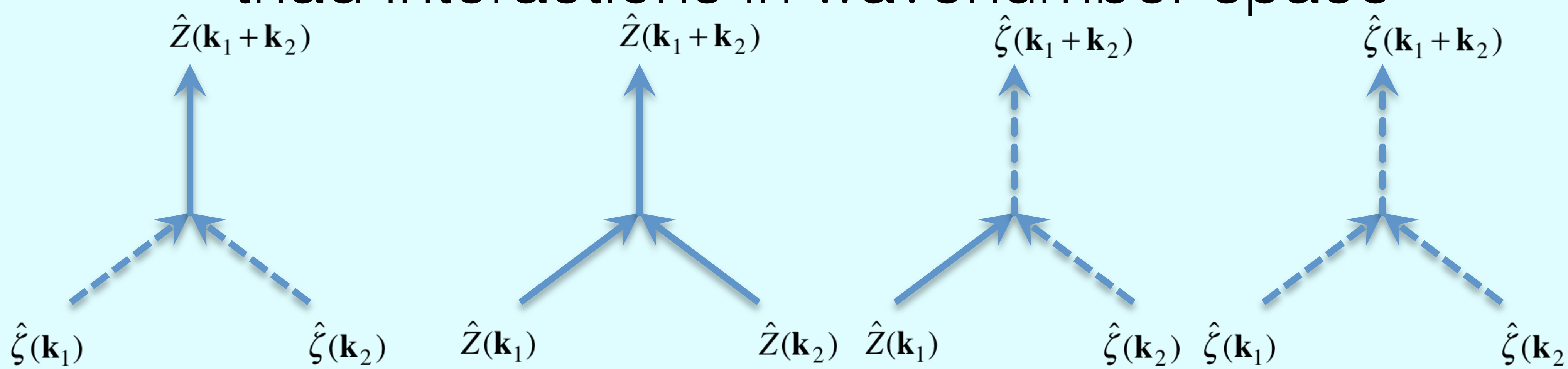
- average over the zonal direction *x*
- Reynolds over an intermediate time scale or length scale
(larger than the time scale or length scale of the turbulent motions
and smaller than the time scale or length scale of mean field)

# NL system

$$\partial_t Z + \mathbf{U}\cdot\nabla Z + \beta V = -\langle \mathbf{u}'\cdot\nabla\zeta'\rangle - rZ$$

$$\partial_t \zeta' = \mathcal{A}(\mathbf{U})\,\zeta' + \langle \mathbf{u}'\cdot\nabla\zeta'\rangle - \mathbf{u}'\cdot\nabla\zeta' + \sqrt{\varepsilon}\,\xi$$

triad interactions in wavenumber space



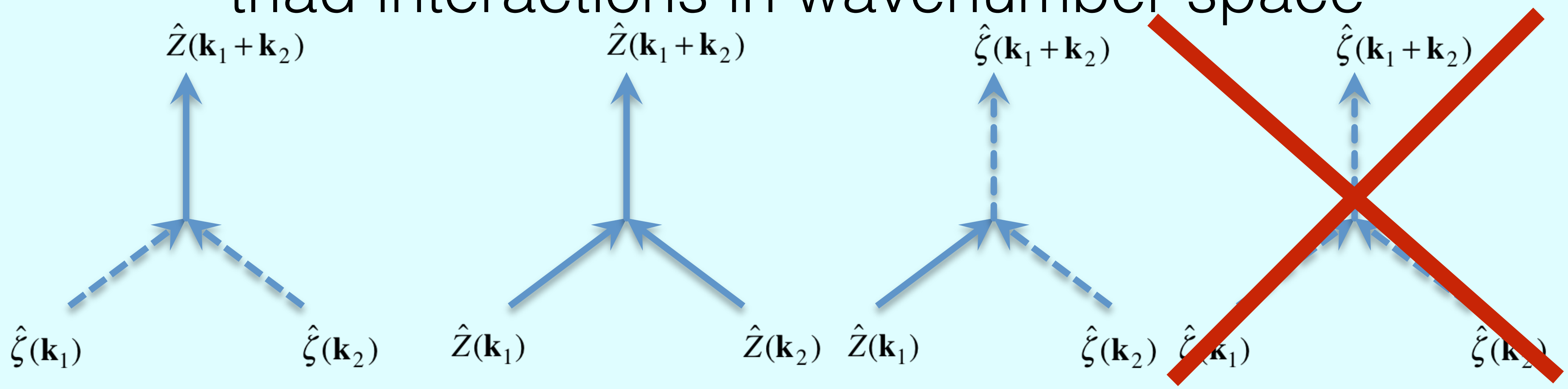
# NL system

restrict nonlinearity by *not* allowing
eddy-eddy → eddy interactions (QL)

$$\partial_t Z + \mathbf{U}\cdot\nabla Z + \beta V = -\left\langle \mathbf{u}'\cdot\nabla\zeta' \right\rangle - rZ$$

$$\partial_t \zeta' = \mathcal{A}(\mathbf{U})\,\zeta' + \cancel{\left\langle \mathbf{u}'\cdot\nabla\zeta' \right\rangle - \mathbf{u}'\cdot\nabla\zeta'} + \sqrt{\varepsilon}\,\xi$$

triad interactions in wavenumber space



# QL system

restrict nonlinearity by *not* allowing
eddy-eddy ➞ eddy interactions (QL)

$$\partial_t Z + \mathbf{U} \cdot \nabla Z + \beta V = - \langle \mathbf{u}' \cdot \nabla \zeta' \rangle - rZ$$

$$\partial_t \zeta' = \mathcal{A}(\mathbf{U})\, \zeta' + \sqrt{\varepsilon}\, \xi$$

QL allows *only* the direct, two-way interaction
of the eddies and the mean flow

QL does NOT include turbulent cascades

QL does NOT include PV mixing

2 out of the
*O*(10) theories

# S3T system

if

$\langle \bullet \rangle$ = ensemble average over forcing realizations

we derive from QL a *closed* system for the evolution of the 1st and 2nd statistical moments of the flow:

$$Z(\mathbf{x},t) = \langle \zeta(\mathbf{x},t) \rangle \quad , \quad C(\mathbf{x}_a,\mathbf{x}_b,t) = \langle \zeta'(\mathbf{x}_a,t)\zeta'(\mathbf{x}_b,t) \rangle$$

1st moment

2nd moment

# S3T system

$$\partial_t Z + \mathbf{U}\cdot\nabla Z + \beta V = \mathcal{R}(C) - rZ$$

$$\partial_t C_{ab} = \left[\mathcal{A}_a(\mathbf{U}) + \mathcal{A}_b(\mathbf{U})\right] C_{ab} + \varepsilon\, Q_{ab}$$

with

$$C_{ab} \overset{\text{def}}{=} C(\mathbf{x}_a, \mathbf{x}_b, t) = \langle \zeta'(\mathbf{x}_a, t)\zeta'(\mathbf{x}_b, t)\rangle$$

$$Q_{ab} \overset{\text{def}}{=} Q(\mathbf{x}_a - \mathbf{x}_b) \longrightarrow$$ the spatial covariance of the statistically homogeneous stochastic forcing

$$\mathcal{R}(C) \overset{\text{def}}{=} -\langle \mathbf{u}'\cdot\nabla\zeta'\rangle = -\nabla\cdot\left[\frac{\hat{\mathbf{z}}}{2}\times\left(\nabla_a\Delta_a^{-1} + \nabla_b\Delta_b^{-1}\right)C_{ab}\right]_{a=b}$$

(the Reynolds stresses are given as a linear function of *C*)

# S3T system

$$\partial_t Z + \mathbf{U} \cdot \nabla Z + \beta V = \mathcal{R}(C) - r Z$$

$$\partial_t C_{ab} = \left[\mathcal{A}_a(\mathbf{U}) + \mathcal{A}_b(\mathbf{U})\right] C_{ab} + \varepsilon \, Q_{ab}$$

Neglect of the eddy-eddy term in NL is equivalent with neglect of third and higher-order statistical moments.

# S3T system

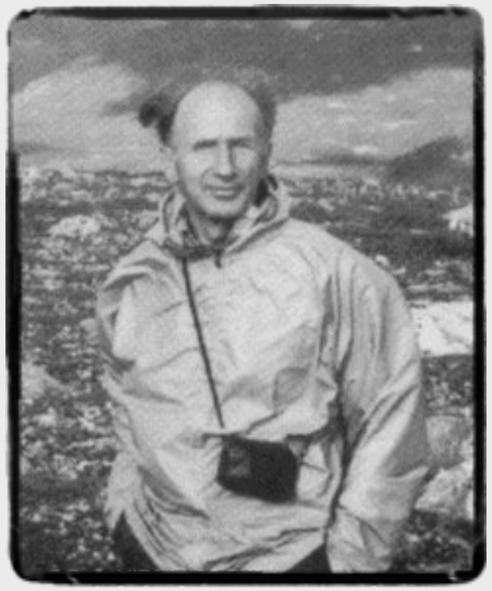
## (the theory)

$$\partial_t Z + \mathbf{U}\cdot\nabla Z + \beta V = \mathcal{R}(C) - rZ$$
$$\partial_t C_{ab} = \left[\mathcal{A}_a(\mathbf{U}) + \mathcal{A}_b(\mathbf{U})\right] C_{ab} + \varepsilon\, Q_{ab}$$

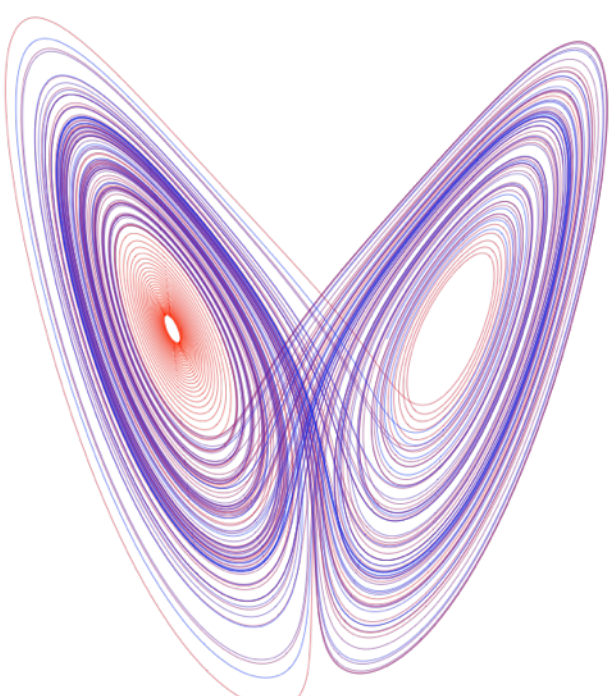
The S3T system

- autonomous
- deterministic (stochasticity has been averaged out)
- admits fixed point solutions consisting of a mean flow and second-order eddy statistics $\left(\mathbf{U}^e(\mathbf{x}), C^e(\mathbf{x}_a, \mathbf{x}_b)\right)$
- allows the study of the stability of such equilibrium solutions

# Lorenz's vision



Ed Lorenz



"More than any other theoretical procedure, numerical integration is also subject to the criticism that it yields little insight into the problem. The computed numbers are not only processed like data but they look like data, and a study of them may be no more enlightening than a study of real meteorological observations. *An alternative procedure which does not suffer this disadvantage consists of deriving a new system of equations whose unknowns are the statistics themselves.*"

The Nature and Theory of the General Circulation of the Atmosphere, by E. N. Lorenz, **1967**

S3T is a first step towards this *new system of equations*

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# for statistically homogeneous forcing there exists *always* a statistically homogeneous S3T equilibrium with no mean flow

$$\mathbf{U}^e = 0 \quad , \quad C^e(\mathbf{x}_a - \mathbf{x}_b) = \frac{\varepsilon\, Q}{2r}$$

(for any $\varepsilon$, $\beta$ and homogeneous $Q$)

zero mean flow + non-zero second-order eddy statistics

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zero mean flow + non-zero second-order eddy statistics

perturbations ($\delta Z,\ \delta C$) about any S3T equilibrium satisfy the linearized S3T equations:

we linearized about a turbulent state!

$$\partial_t \delta Z = \mathcal{A}(\mathbf{U}^e)\,\delta Z + \mathcal{R}(\delta C)$$

$$\partial_t \delta C_{ab} = [\mathcal{A}_a(\mathbf{U}^e) + \mathcal{A}_b(\mathbf{U}^e)]\,\delta C_{ab} + (\delta\mathcal{A}_a + \delta\mathcal{A}_b)C^e_{ab}$$

$$\delta\mathcal{A} \overset{\text{def}}{=} \mathcal{A}(\mathbf{U}^e + \delta\mathbf{U}) - \mathcal{A}(\mathbf{U}^e)$$

eigenanalysis of this system determines the stability of $\big(\mathbf{U}^e(\mathbf{x}), C^e(\mathbf{x}_a, \mathbf{x}_b)\big)$

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perturbations ($\delta Z,\ \delta C$) about any S3T equilibrium satisfy the linearized S3T equations:

hydrodynamic stability

$$\partial_t \delta Z = \mathcal{A}(\mathbf{U}^e)\,\delta Z + \mathcal{R}(\delta C)$$

we linearized about a turbulent state!

$$\partial_t \delta C_{ab} = \left[\mathcal{A}_a(\mathbf{U}^e) + \mathcal{A}_b(\mathbf{U}^e)\right]\delta C_{ab} + (\delta\mathcal{A}_a + \delta\mathcal{A}_b) C^e_{ab}$$

$$\delta\mathcal{A} \overset{\text{def}}{=} \mathcal{A}(\mathbf{U}^e + \delta\mathbf{U}) - \mathcal{A}(\mathbf{U}^e)$$

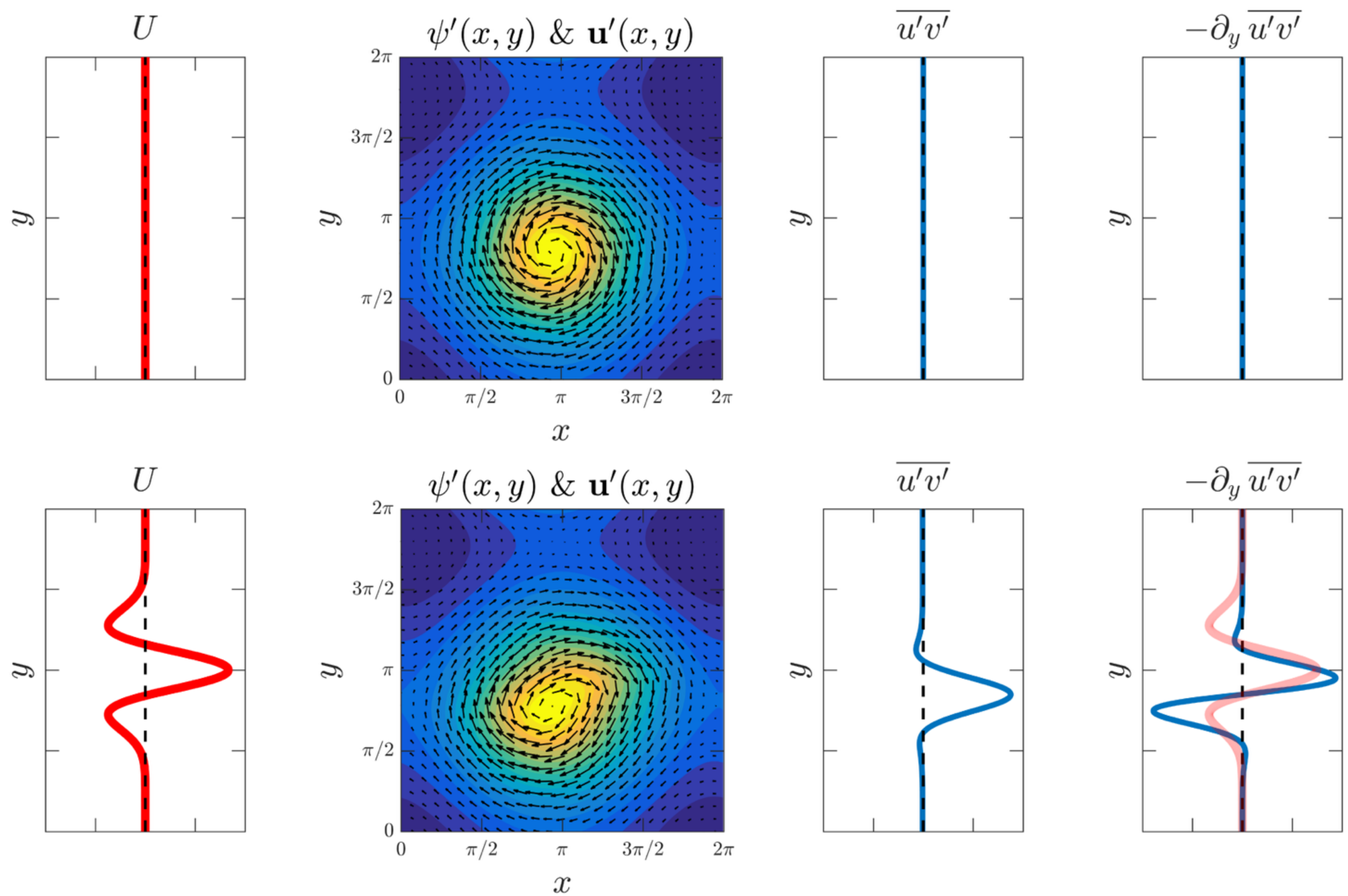
eigenanalysis of this system determines the stability of $\left(\mathbf{U}^e(\mathbf{x}), C^e(\mathbf{x}_a, \mathbf{x}_b)\right)$

# proof of concept

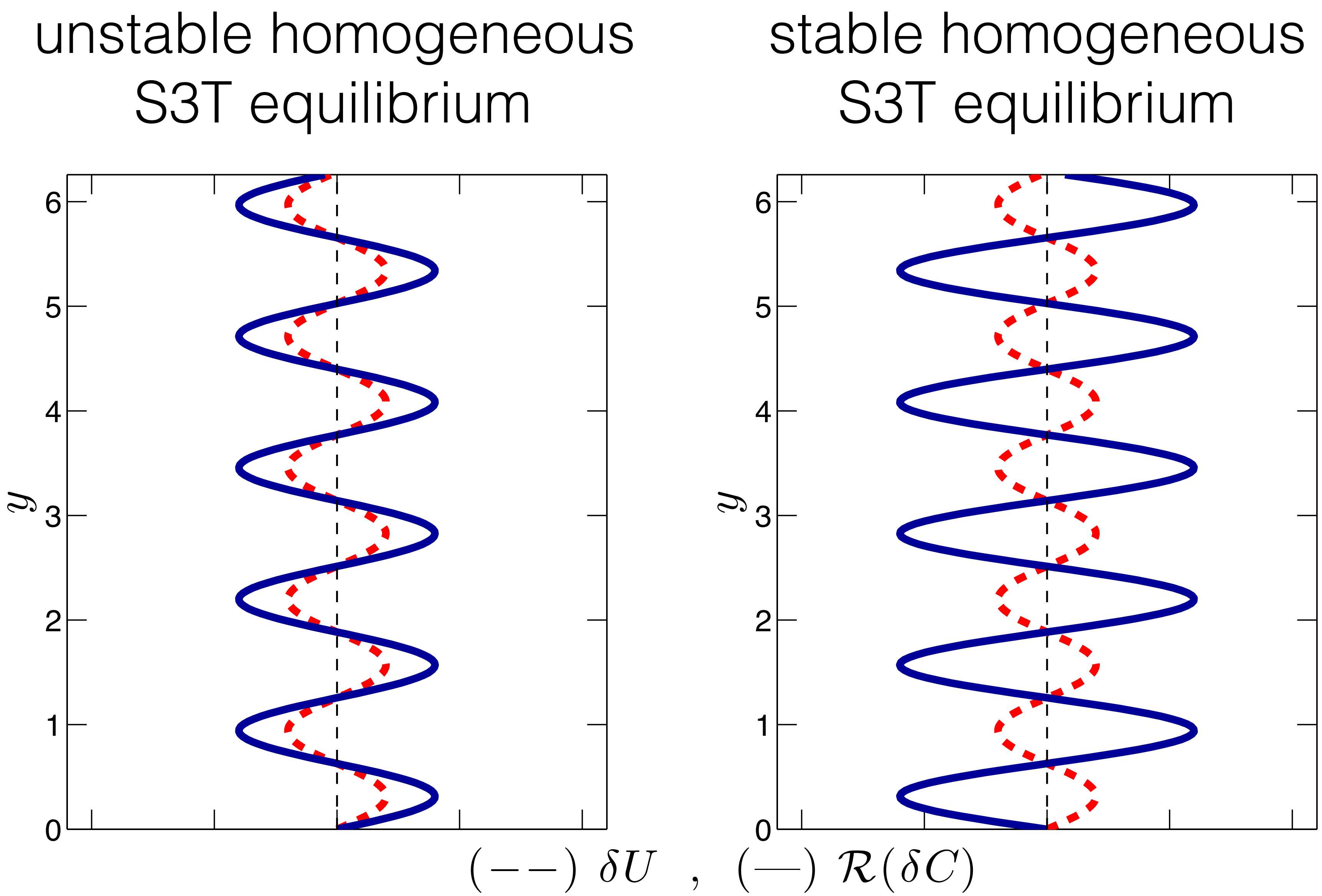
## how does a zero jet state become unstable?

for certain parameters eddies have the tendency to reinforce mean flow inhomogeneities (even if mean flow is infinitesimal!)

$$\partial_t \delta Z = \mathcal{A}(\mathbf{U}^e)\,\delta Z + \boxed{\mathcal{R}(\delta C)}$$



# the Reynolds stresses will act so as to reinforce or diminish the infinitesimal mean flow

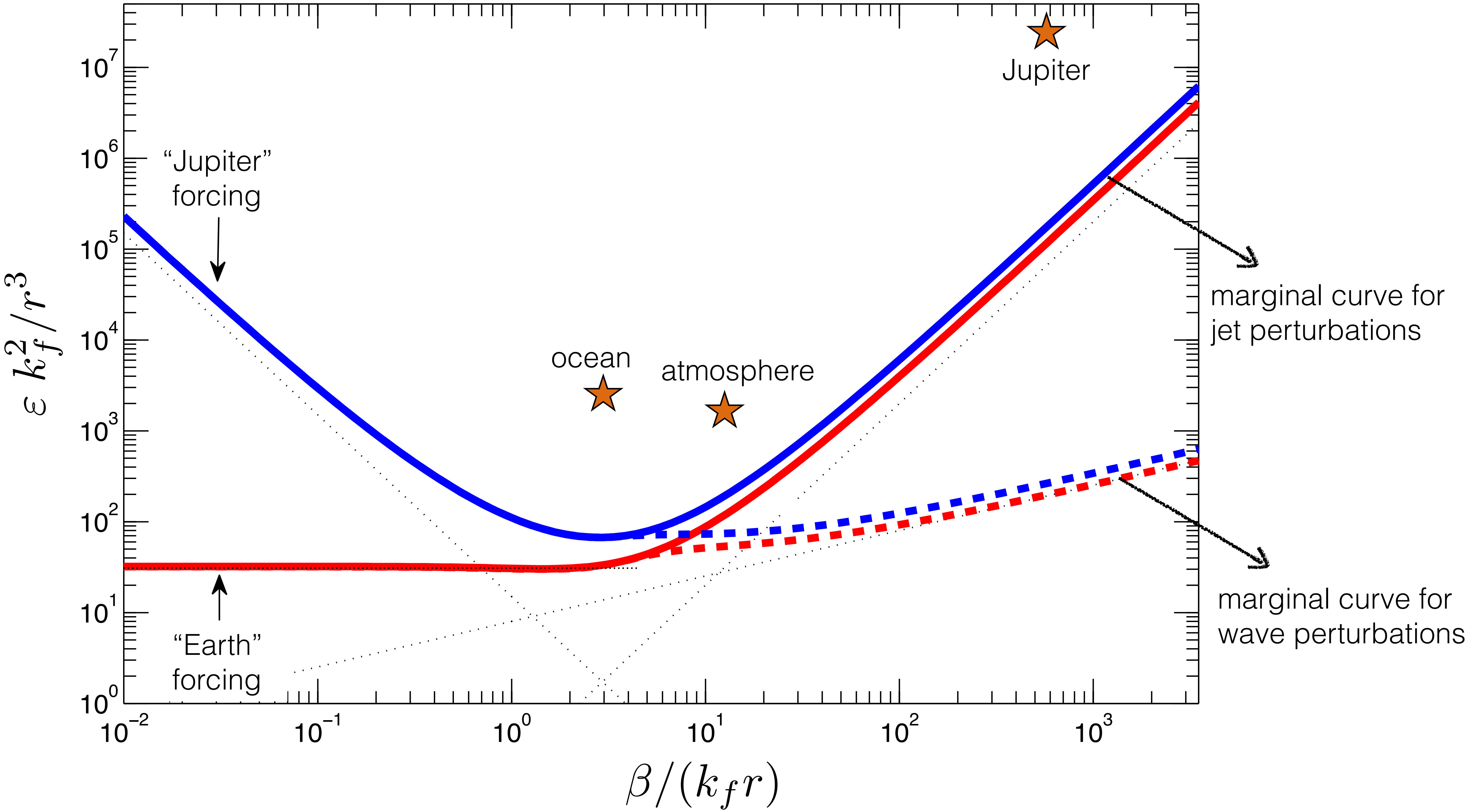


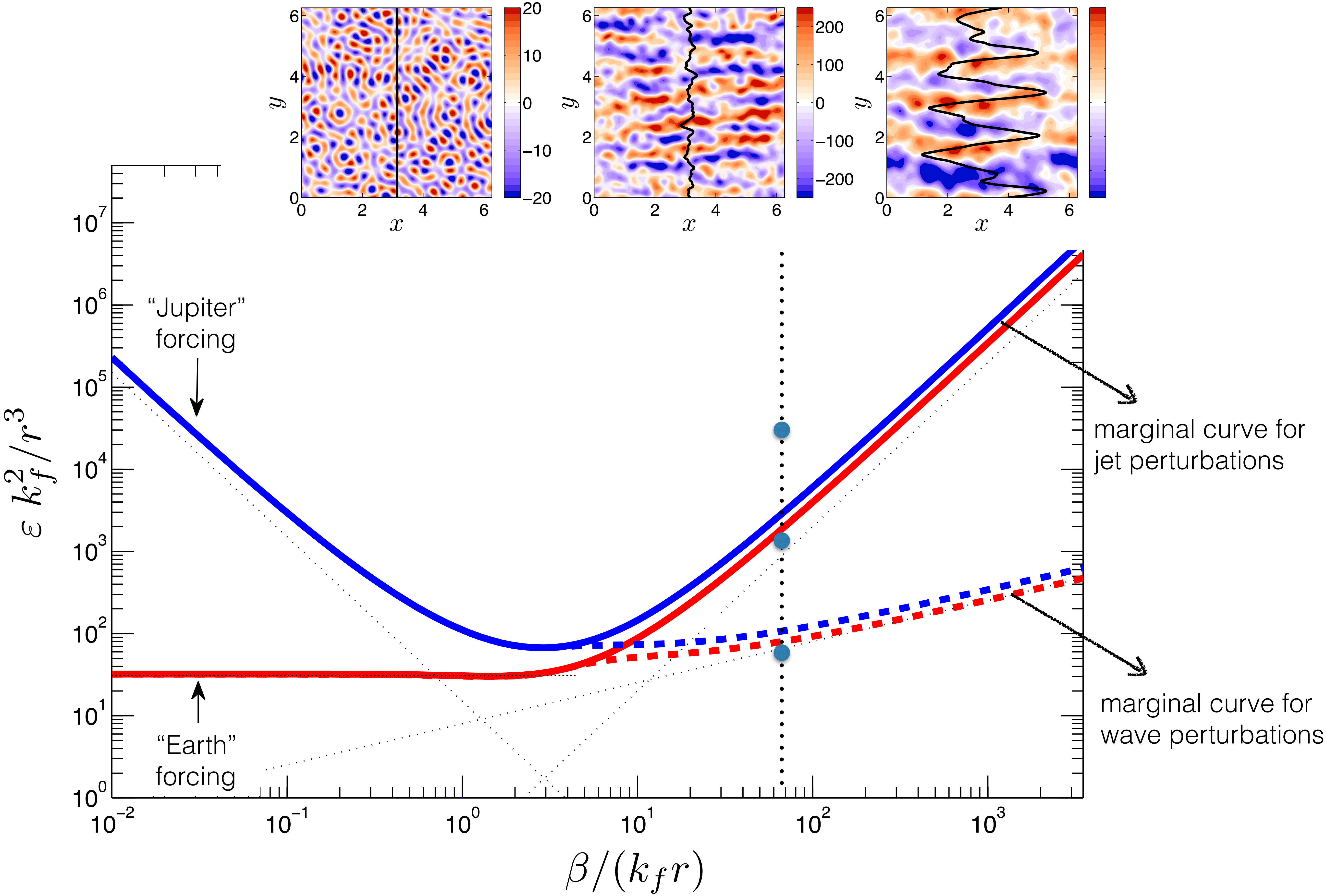

turbulence acts as

anti-diffusion

diffusion

# Marginal curve for S3T instability of the homogeneous turbulent state





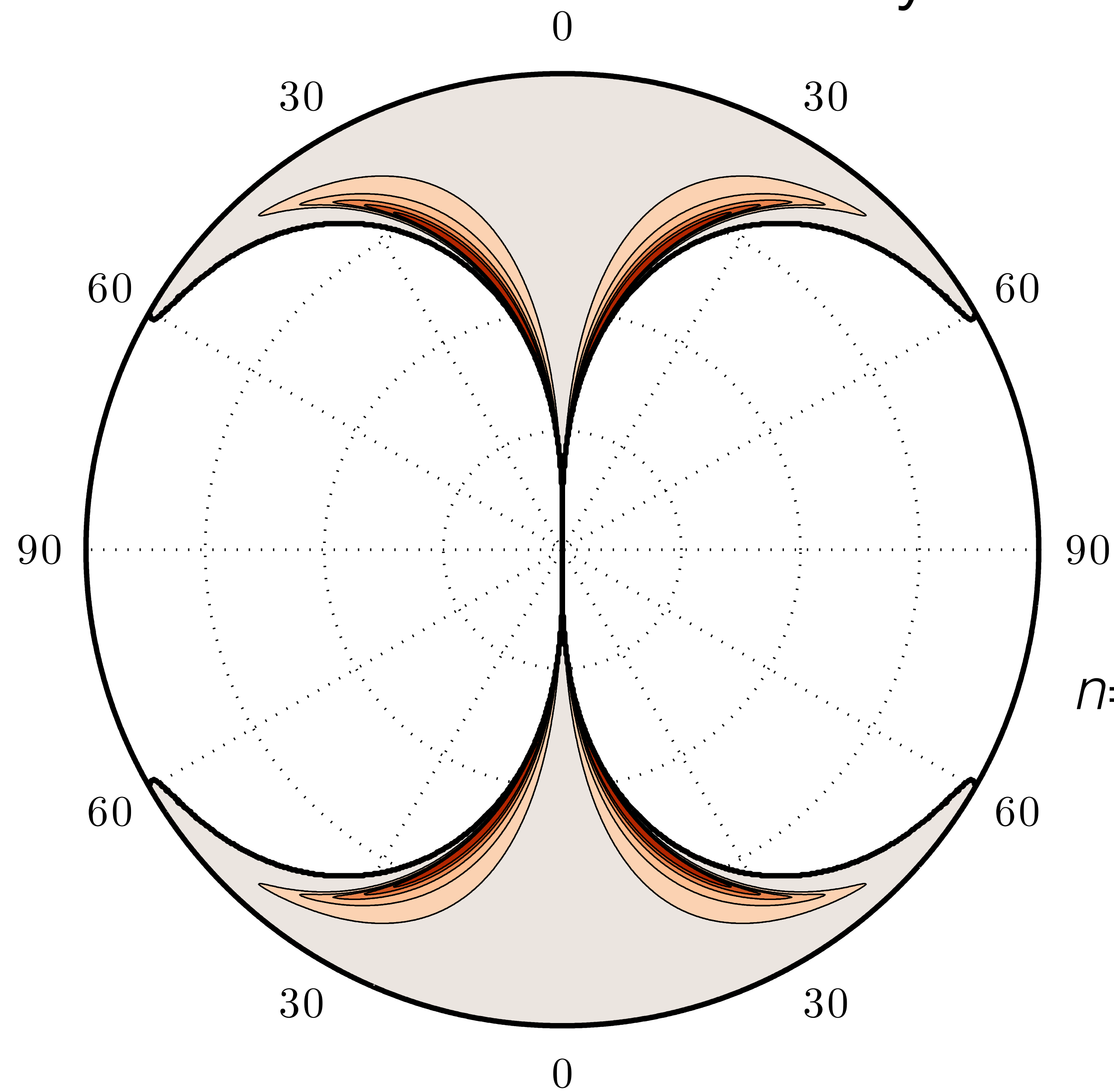
“Jupiter” forcing
“Earth” forcing
marginal curve for jet perturbations
marginal curve for wave perturbations
$\varepsilon\, k_f^2/r^3$
$\beta/(k_f r)$
$x$
$y$

for which perturbation mean flow wavevectors ($n_x$,$n_y$) does S3T predicts that Reynolds stresses will act anti-diffusively?

$\beta/(k_f r) = 200$

$\phi$=atan($n_x$/$n_y$)

$\phi$=0 ⟶ jets




$n$=$k_f$

dumbbell (?)

(completely different exegesis — S3T does not include turbulent cascades!)

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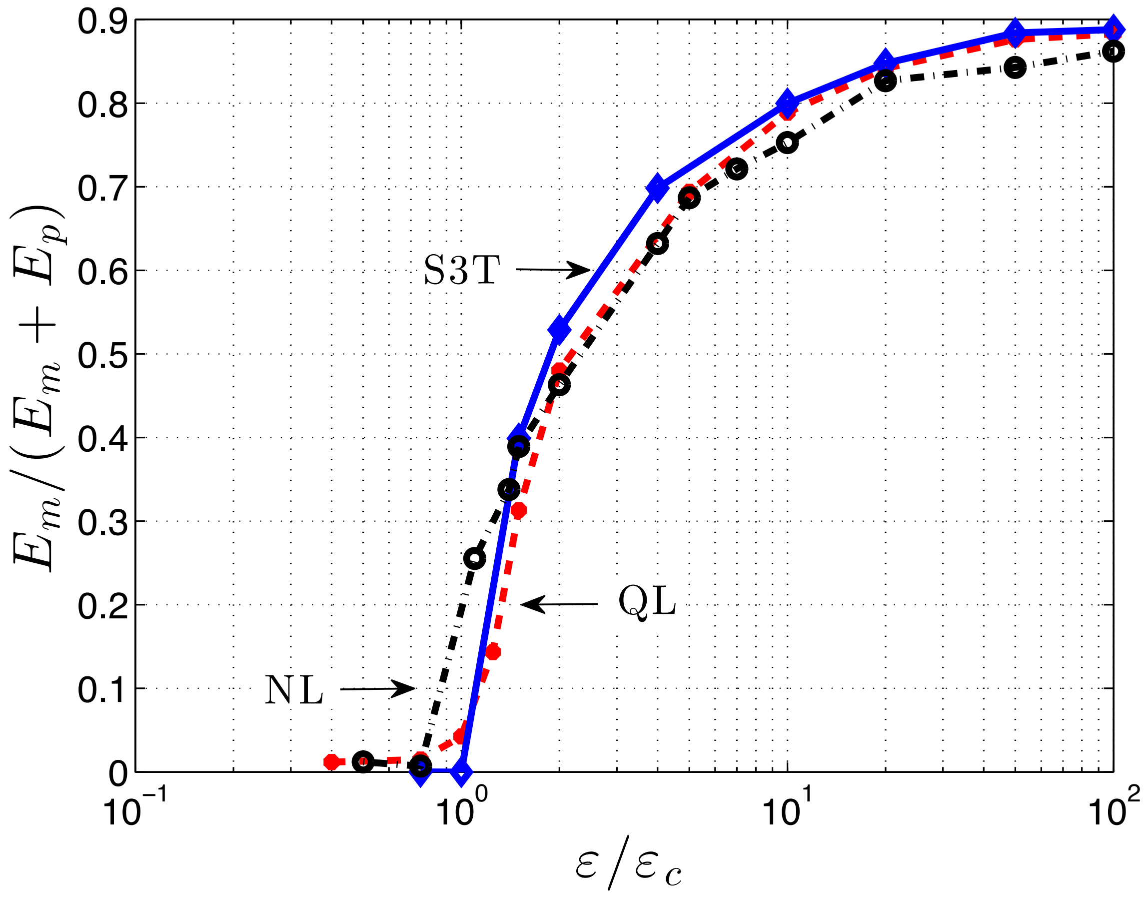
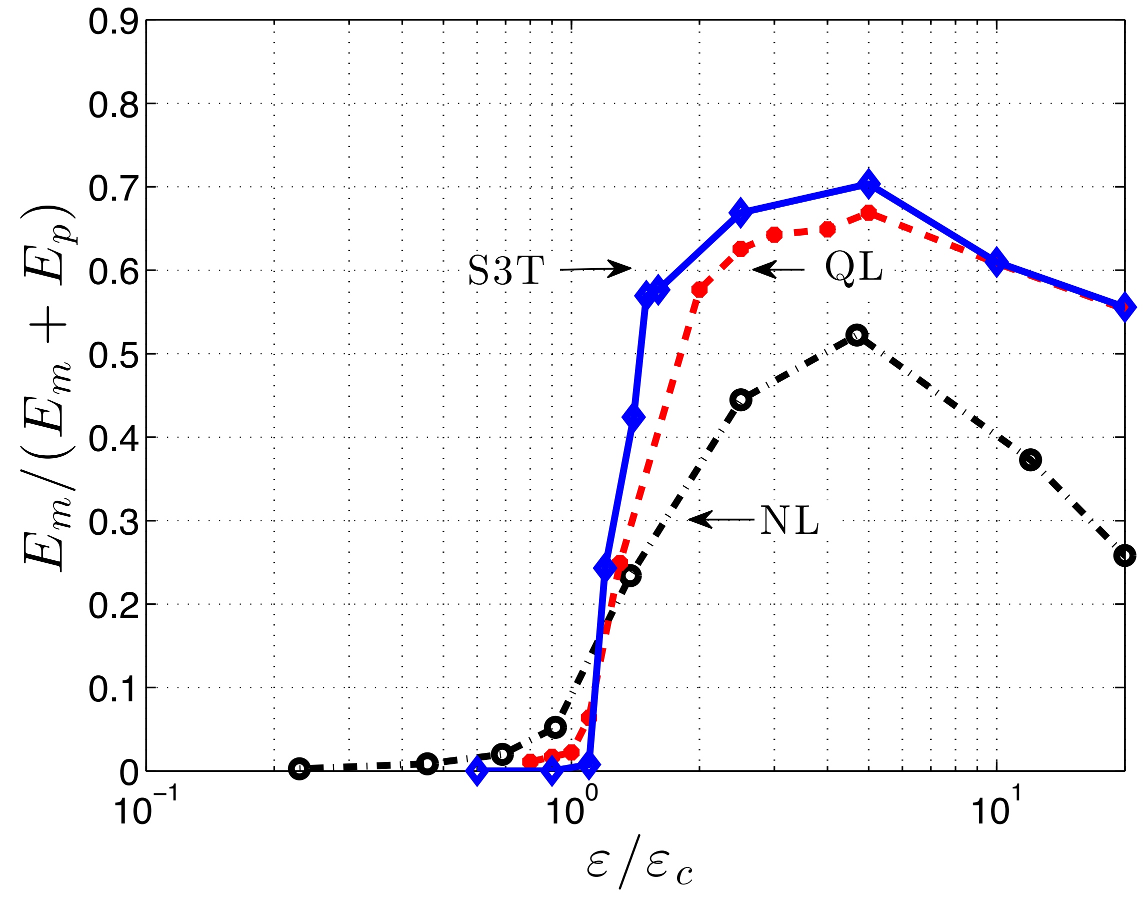
(bit tired? the rest will go fast — don't worry)

# S3T predictions for jet formation and equilibration at finite amplitude
(best case)

anisotropic forcing
[≈Earth]



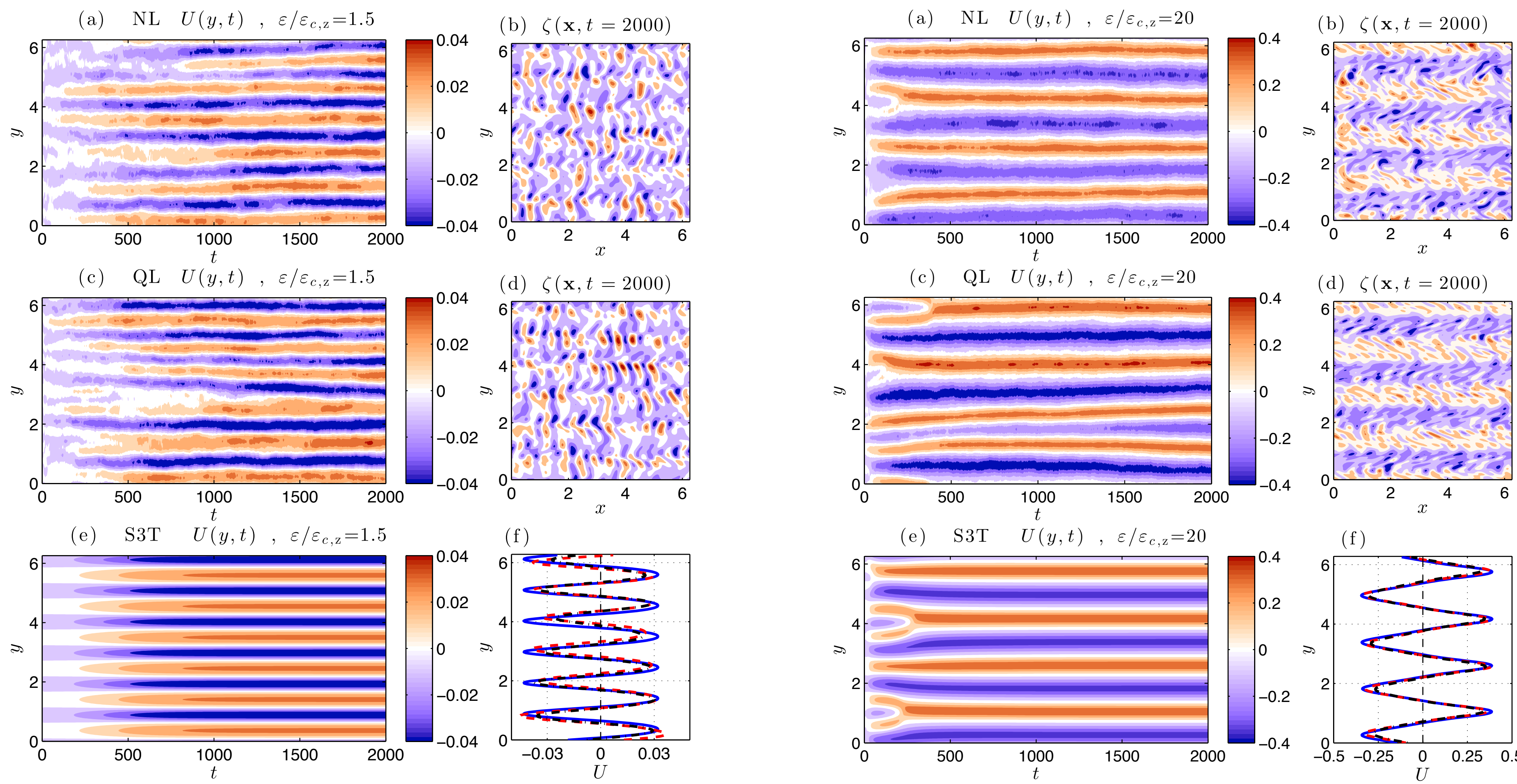

isotropic forcing
[≈Jupiter]



# S3T predictions for jet formation and equilibration at finite amplitude

anisotropic forcing

$\varepsilon/\varepsilon_{c,z}$ = 1.5 [≈Earth] $\varepsilon/\varepsilon_{c,z}$ = 20



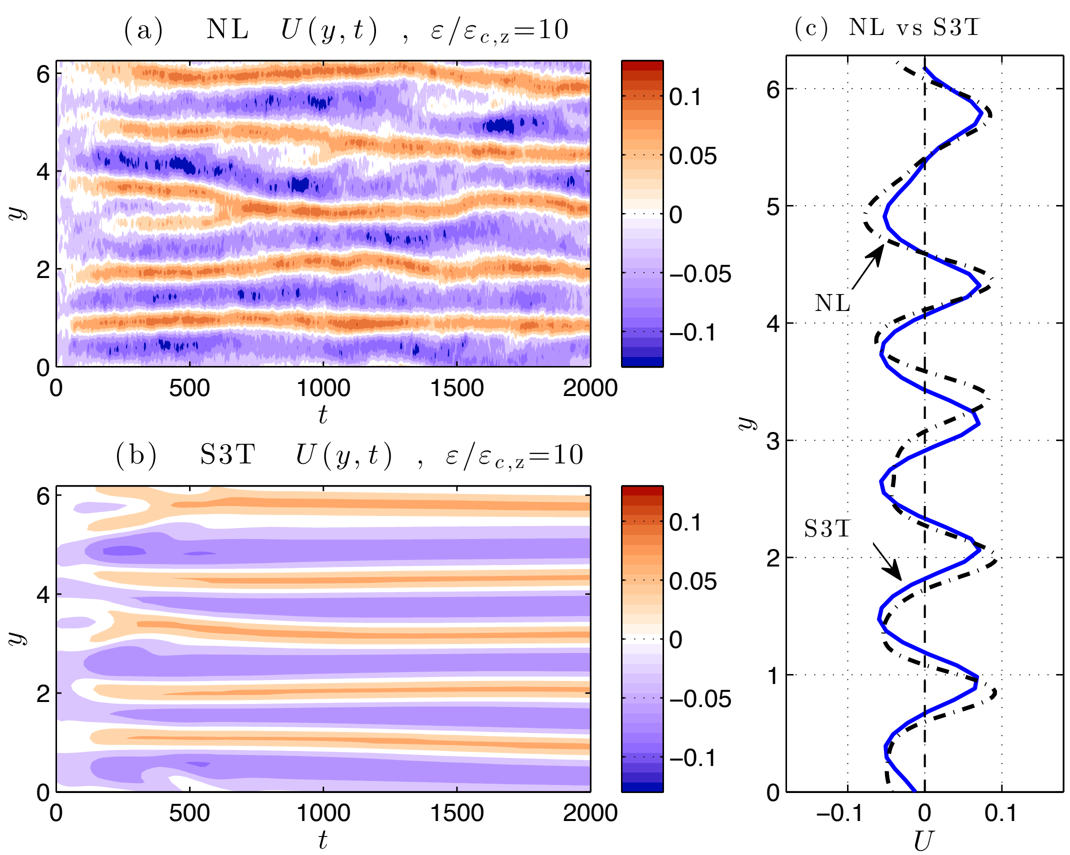

statistical instabilities that are predicted by S3T show up in single NL/QL realizations of the flow

emergent instabilities grow and reach finite amplitude

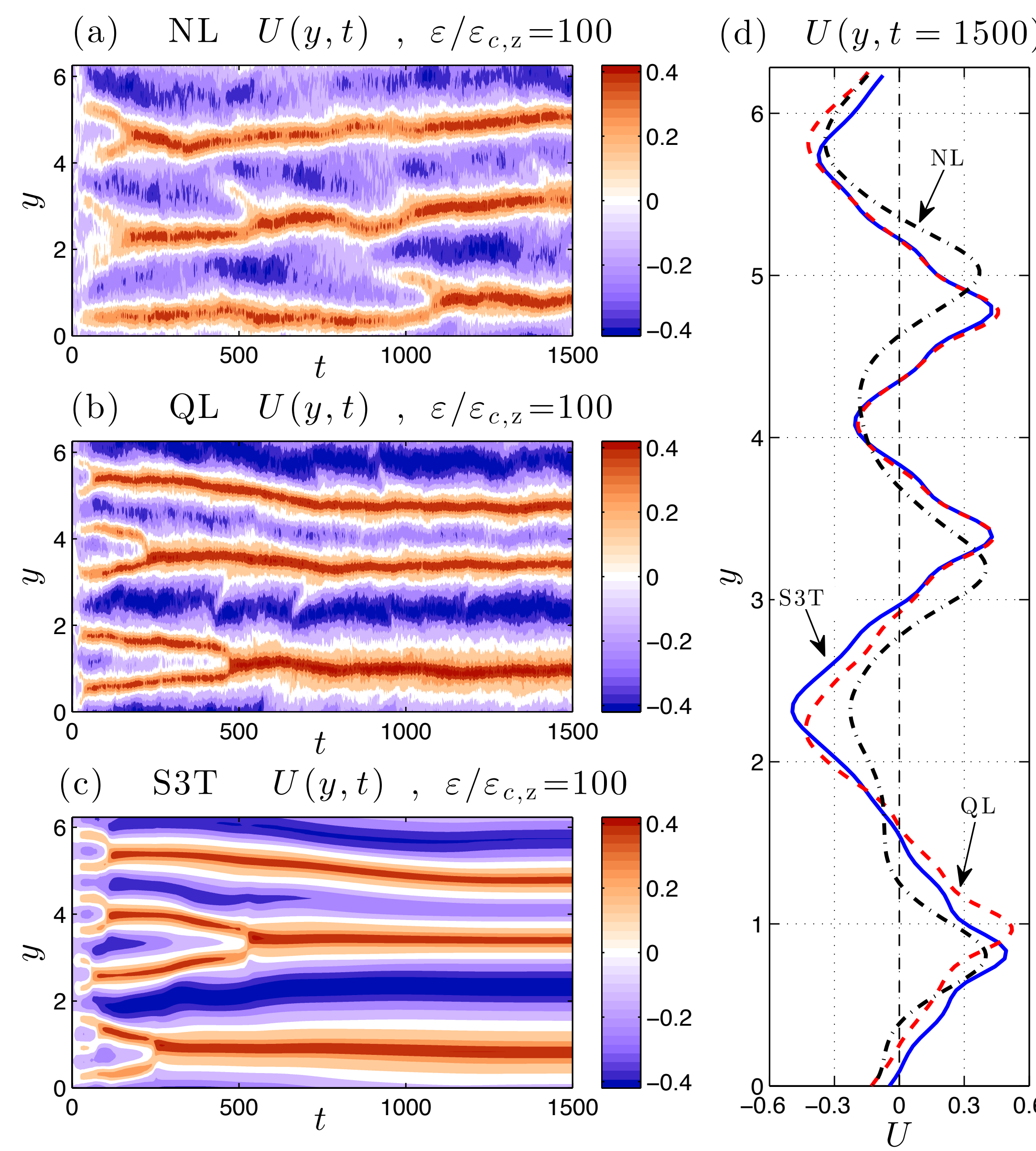
# S3T predictions for jet formation and equilibration at finite amplitude

isotropic forcing
[≈Jupiter]

$\varepsilon/\varepsilon_{c,z}$ = 10




$\varepsilon/\varepsilon_{c,z}$ = 100



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# Zonal jet S3T equilibria

$$\mathbf{U}^e(\mathbf{x}) = \Big(U^e(y), 0\Big) \quad , \quad C^e(x_a - x_b, y_a, y_b)$$

Developed numerical methods for

i) determining such equilibria with great accuracy and

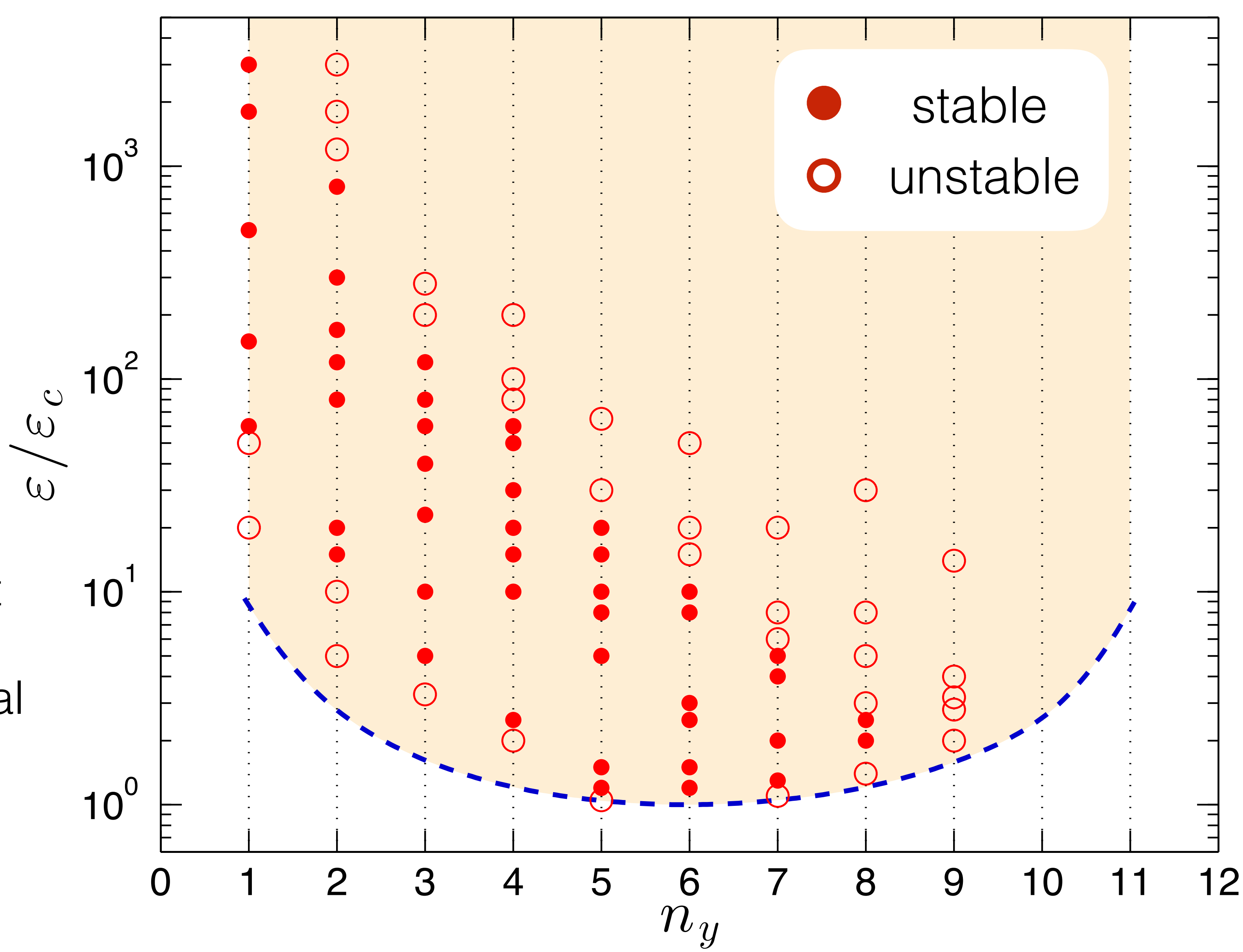
ii) studying their S3T stability

[don't forget that $N$ points in each *x,y* direction
result to a state vector of $O(N^4)$ !]

# Stability of zonal jet S3T equilibria to zonal jet perturbations

Stability analysis of inhomogeneous turbulent states with zonal jets predicts:

- existence of multiple equilibria and their domain of attraction
- merging of jets as $\varepsilon$ increases
- finite amplitude equilibration at small supercriticality is described through the universal Eckhaus instability of the G-L amplitude equation

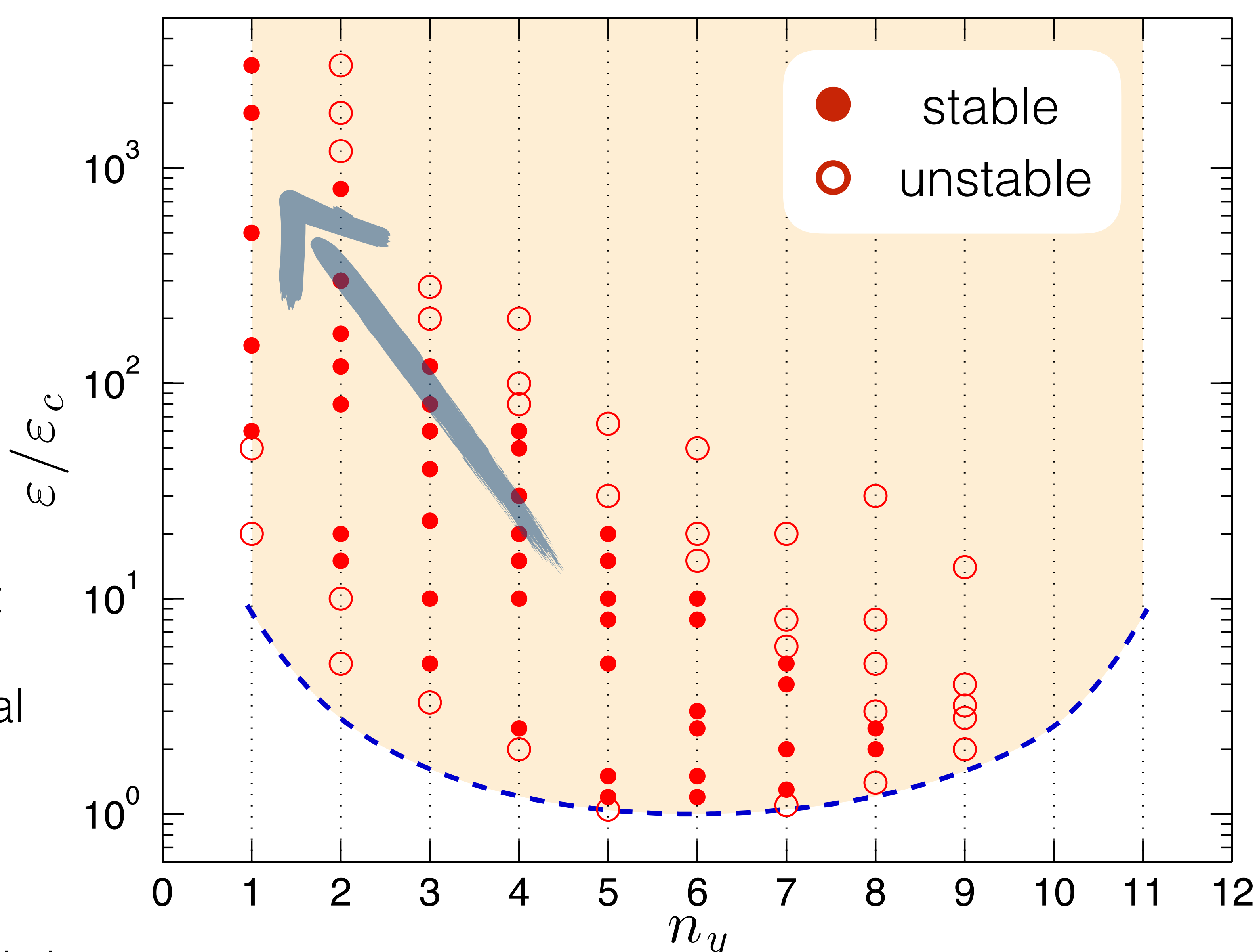



(= number of jets for a $2\pi \times 2\pi$ channel)

# Stability of zonal jet S3T equilibria to zonal jet perturbations

Stability analysis of inhomogeneous turbulent states with zonal jets predicts:

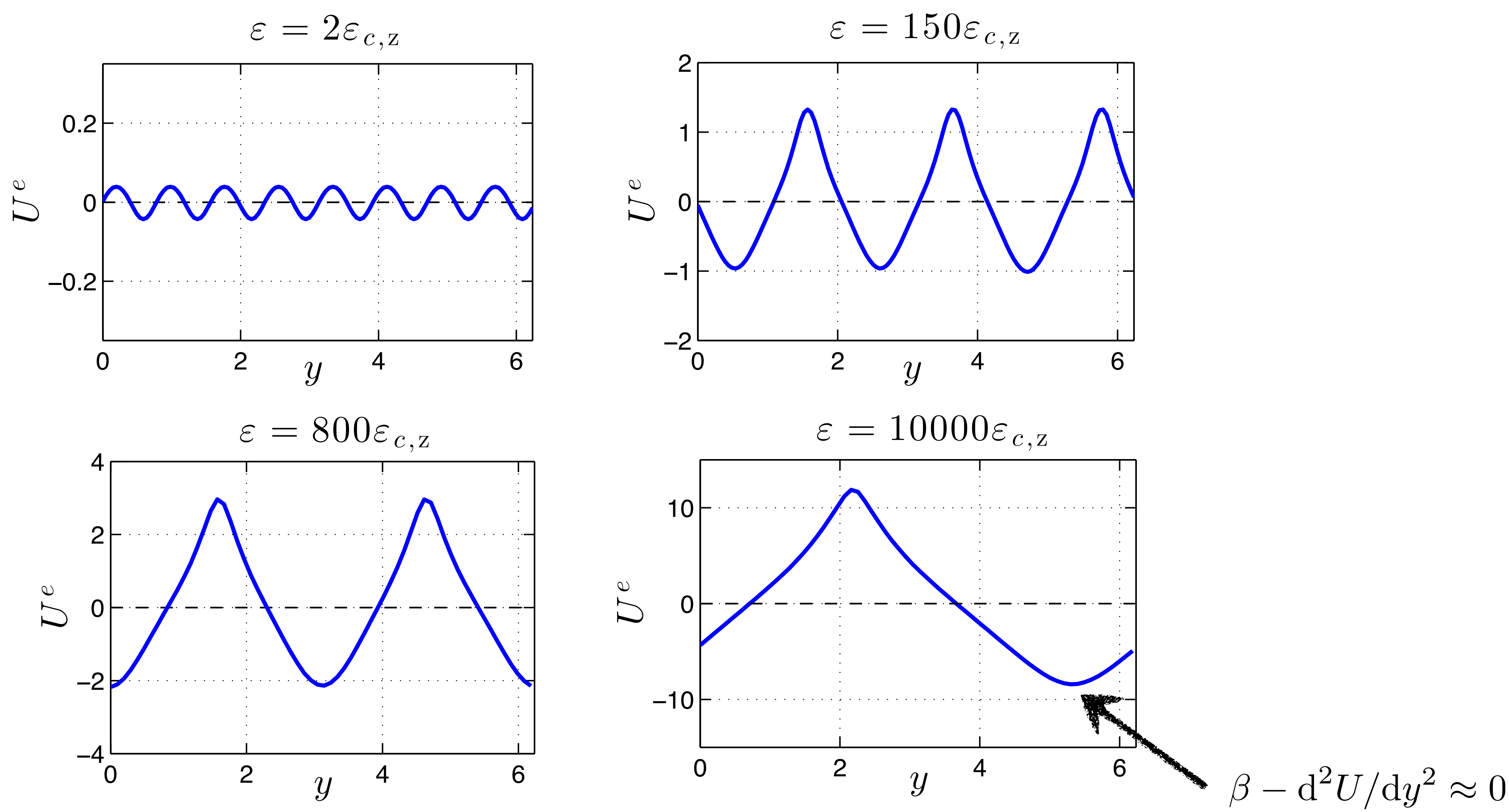
- existence of multiple equilibria and their domain of attraction
- merging of jets as $\varepsilon$ increases
- finite amplitude equilibration at small supercriticality is described through the universal Eckhaus instability of the G-L amplitude equation




(= number of jets for a $2\pi \times 2\pi$ channel)

For higher energy input rates equilibria become S3T unstable and move towards the left of the diagram

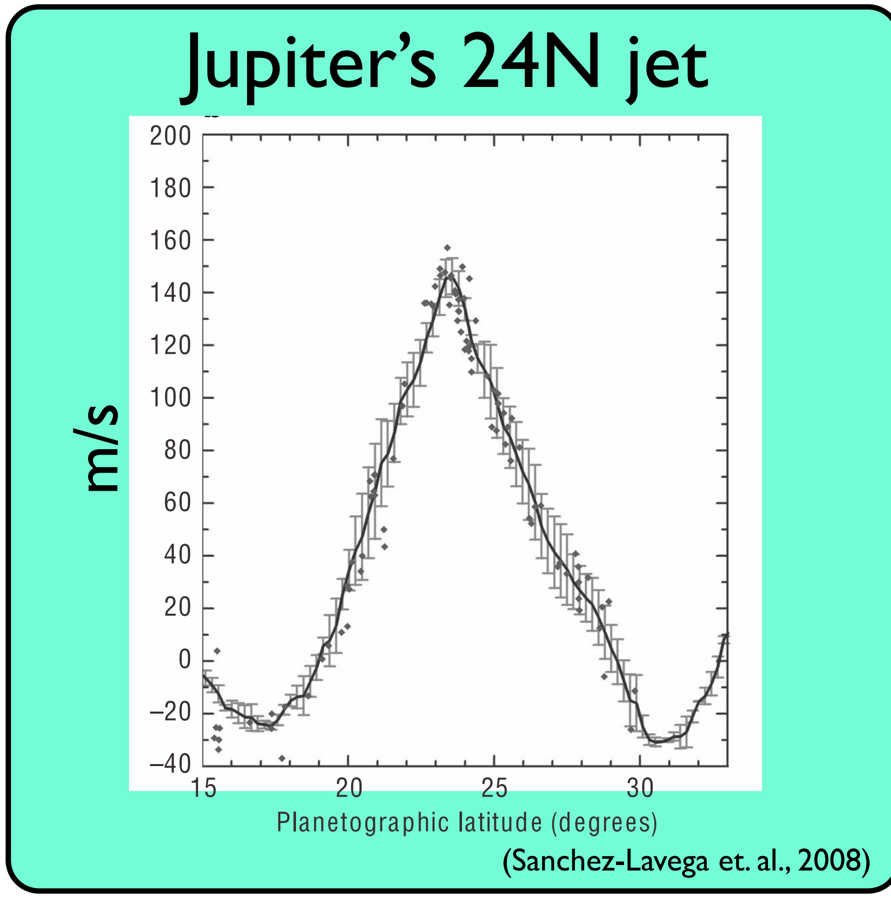
# Structure of zonal jet S3T equilibria

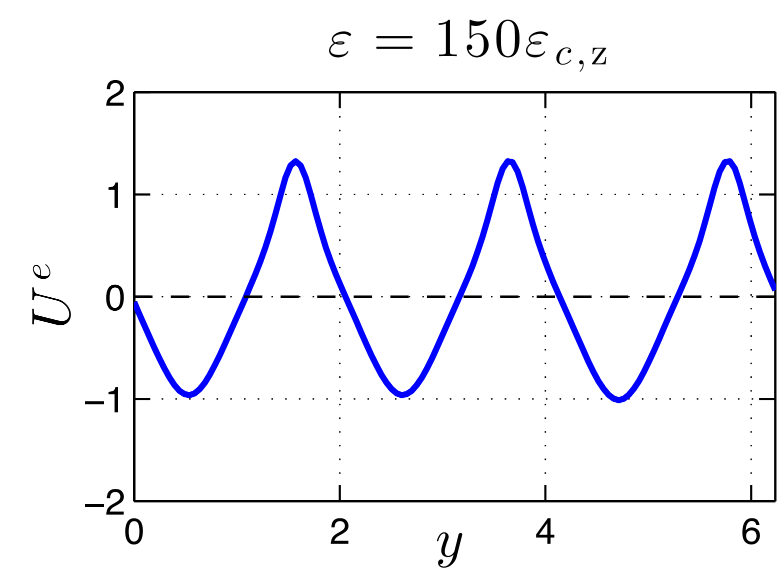


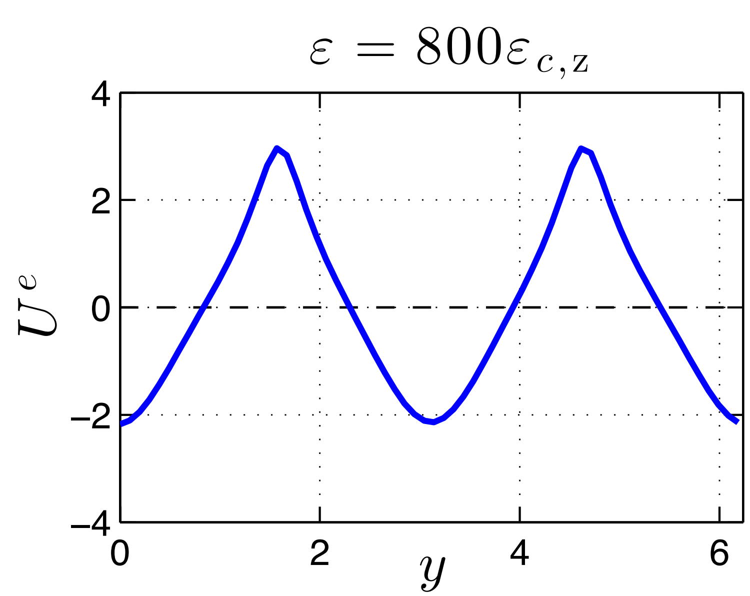

Rayleigh-Kuo
hydrodynamic stability
criterion

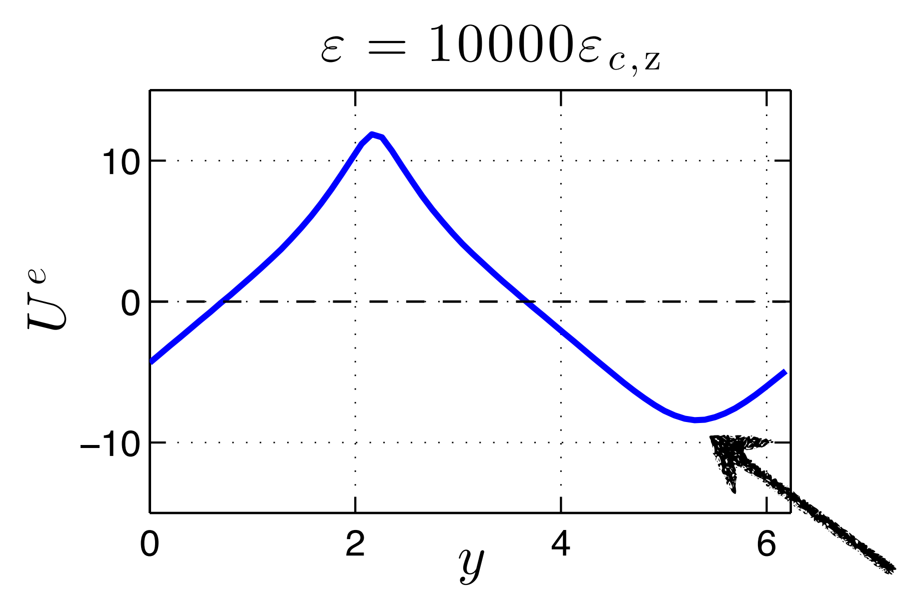
The jet structure therefore is *not* a result of cascades
nor nonlinear PV mixing (PV staircases)

# al jet S3T equilibria




$\beta - \mathrm{d}^2U/\mathrm{d}y^2 \approx 0$

Rayleigh-Kuo
hydrodynamic stability
criterion

The jet structure therefore is *not* a result of cascades
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# structure of talk

- introduction to the physical problem
- formulation of the theory (S3T)
- the homogeneous turbulent state and its stability
- comparison of S3T predictions with direct numerical simulations and verification of the theory
- stability of inhomogeneous turbulent states & relation with jet mergers
- summary

# Conclusions

- S3T generalizes the hydrodynamic stability of Rayleigh and allow us to study the stability of turbulent flows
- S3T makes detailed analytical predictions for the emergence and form of large-scale structure in planetary turbulence
- S3T predicts that the transition from a homogeneous to an inhomogeneous turbulent state occurs through a bifurcation of the statistical state dynamics (homogeneous turbulence is unstable)
- S3T predicts the equilibrated structure of the emergent large-scale flow
- The stability of inhomogeneous turbulent equilibria (e.g. the climate state of the Earth or Jupiter) can be studied within S3T framework.
- Lorenz was right — this *new system of equations* provides more insight than numerical simulations


thank you