Topographic beta-plane turbulence and form stress

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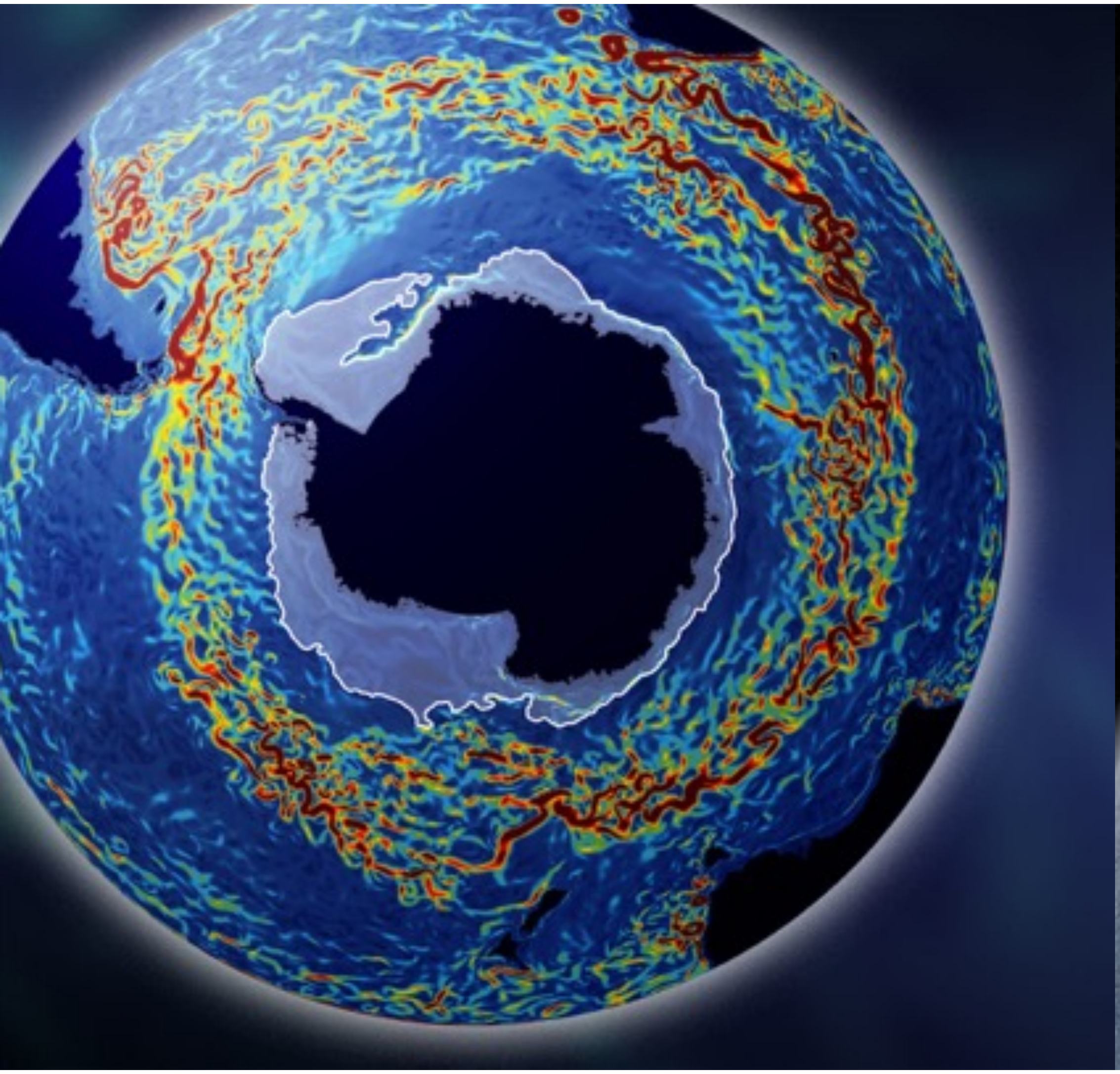
(work in progress)

19 June 2016
WHOI GFD

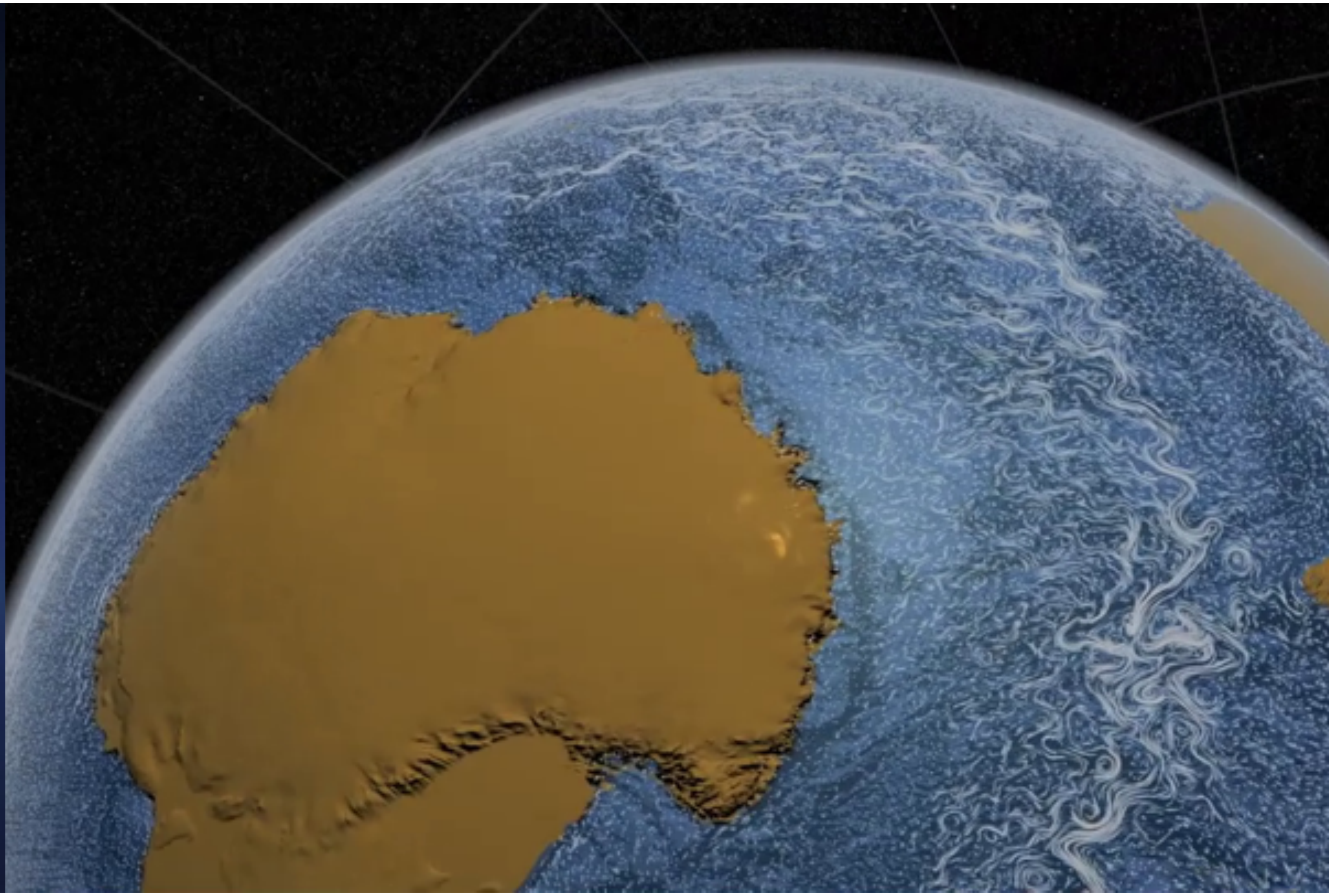
how does the bottom topography of the ocean affect the large-scale zonal oceanic currents?

(e.g. the Antarctic Circumpolar Current)

Antarctic Circumpolar Current (ACC)



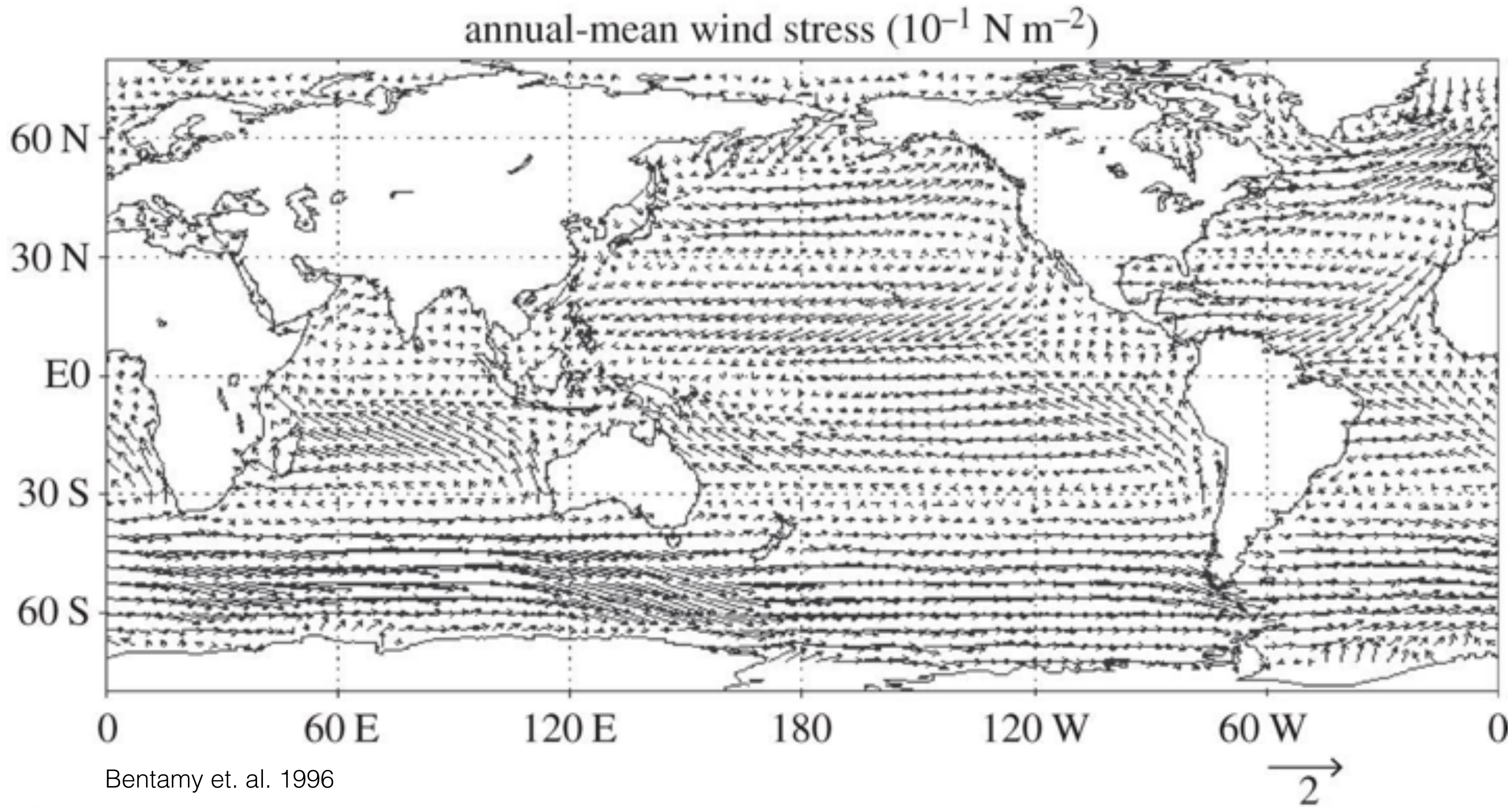
Southern Ocean State Estimate
UC San Diego



NASA/Goddard Space Flight Center

state estimates
(computer simulations
constrained by observations)

momentum is imparted to the ocean by winds

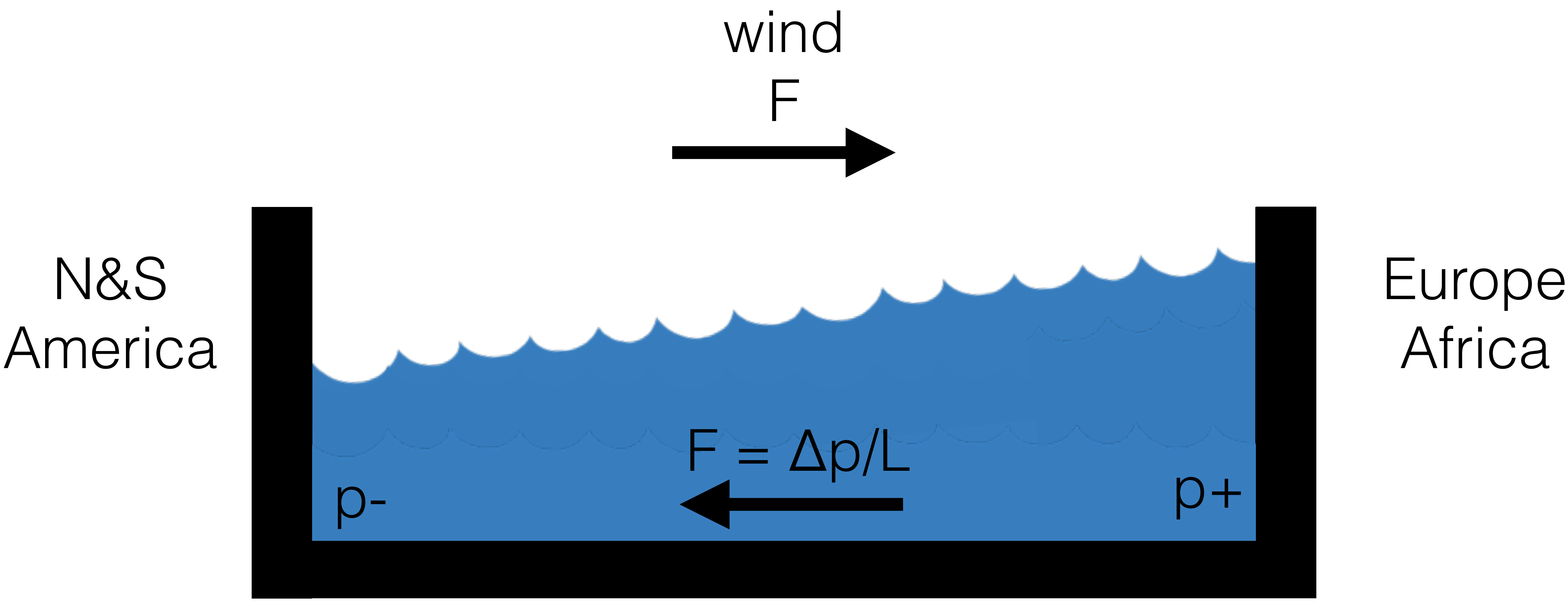


Bentamy et. al. 1996

winds are (on average) easterlies or westerlies

how does the force applied to the oceans by the winds balance?

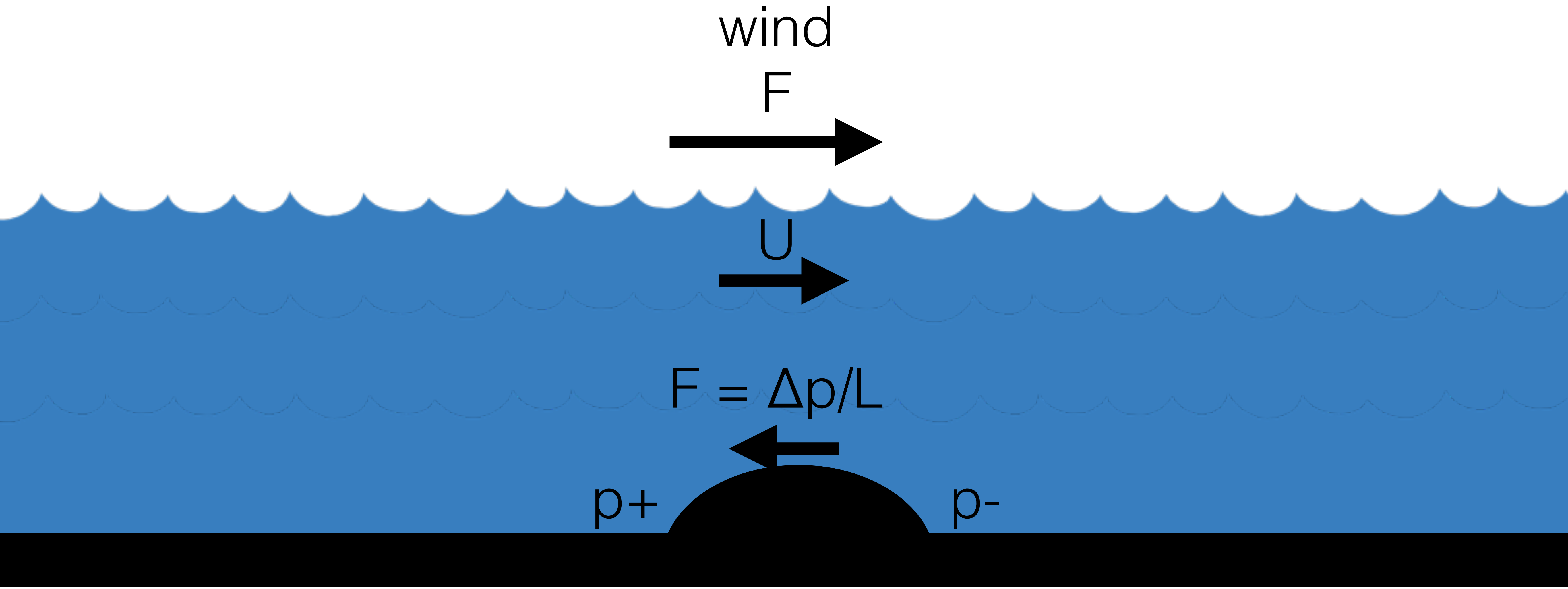
ocean with continental boundaries
(e.g. Atlantic)



the surface of the ocean tilts and creates
an east-west pressure gradients that
mostly balances the momentum input

(the ocean leans onto the eastern coast)

ocean without continental boundaries (e.g. Southern Ocean)



the flow over ocean ridges creates pressure differences that counterbalance the momentum input

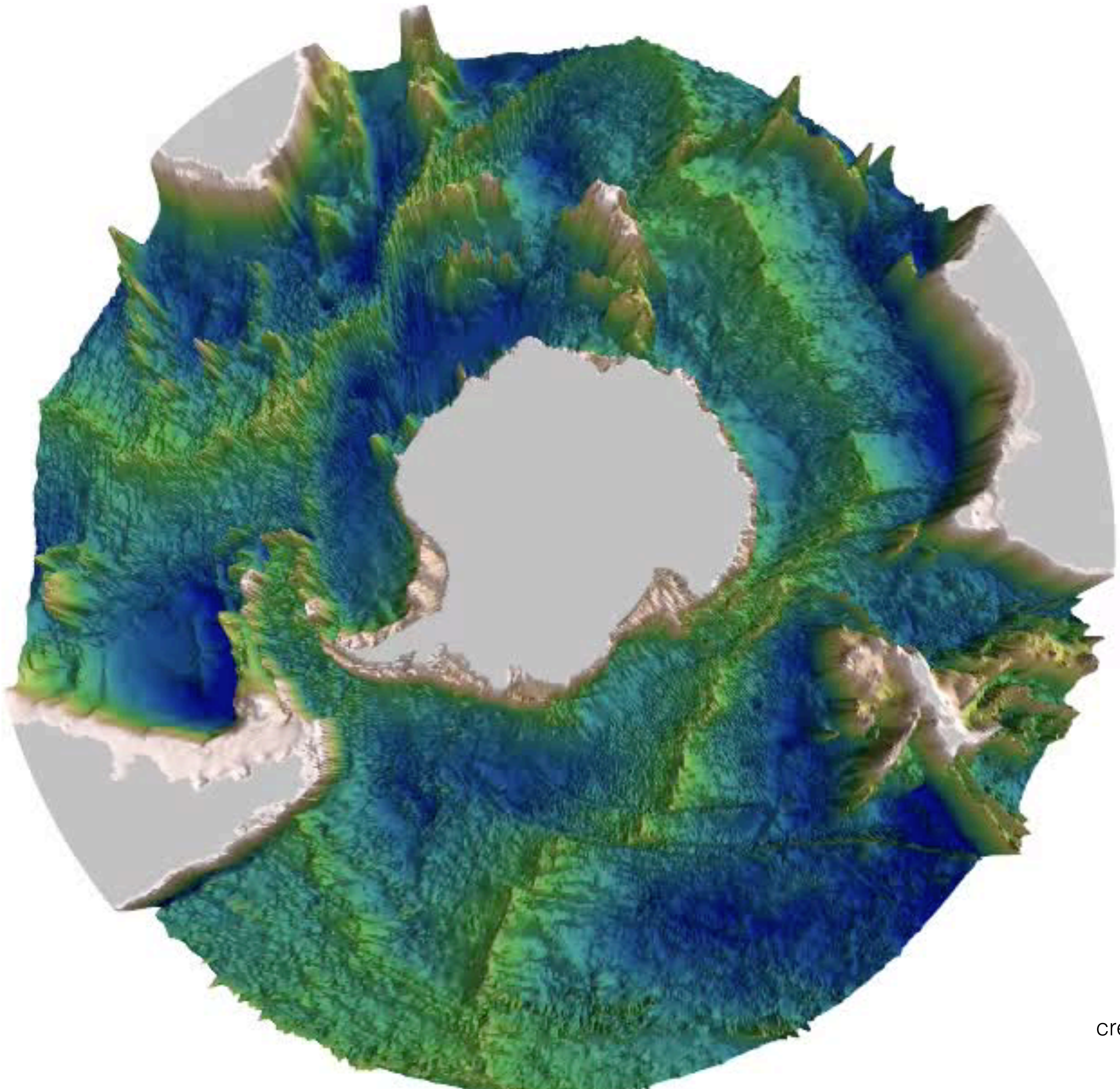
"I don't know why I don't care about the bottom of the ocean, but I don't."

initially work didn’t focus on the role of
the bottom topography

in a seminal paper Munk & Palmen 1951
with a back-of-the-envelope calculation estimated that:

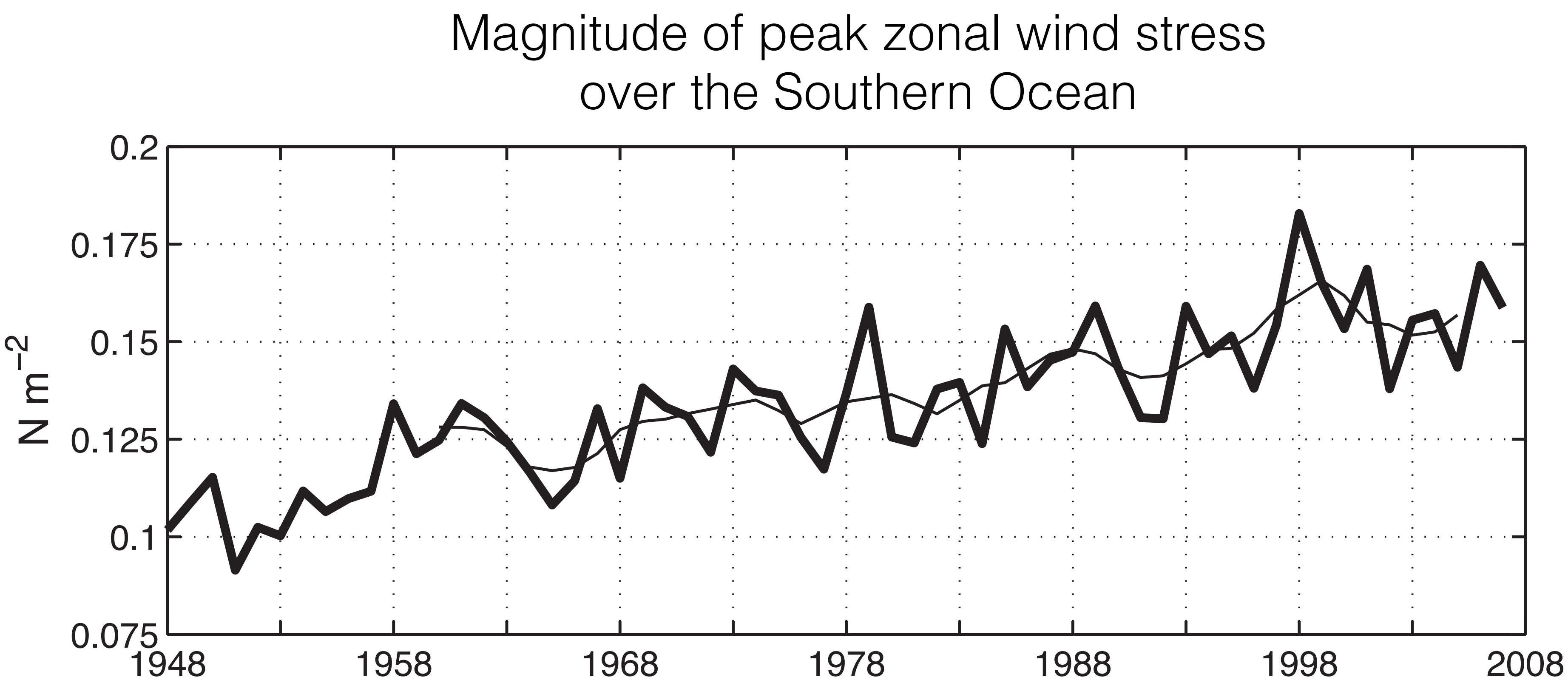
if the bottom of the Southern Ocean was flat
then the ACC should be 10-20 times stronger than observed!

topography in the Southern Ocean



credit: V. Tamsitt, Scripps, UCSD

yet some more motivation…

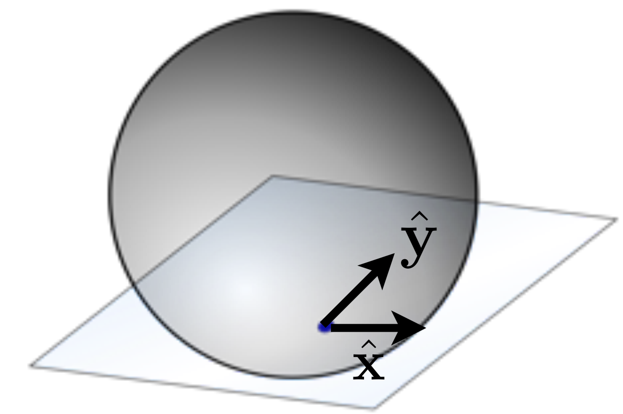


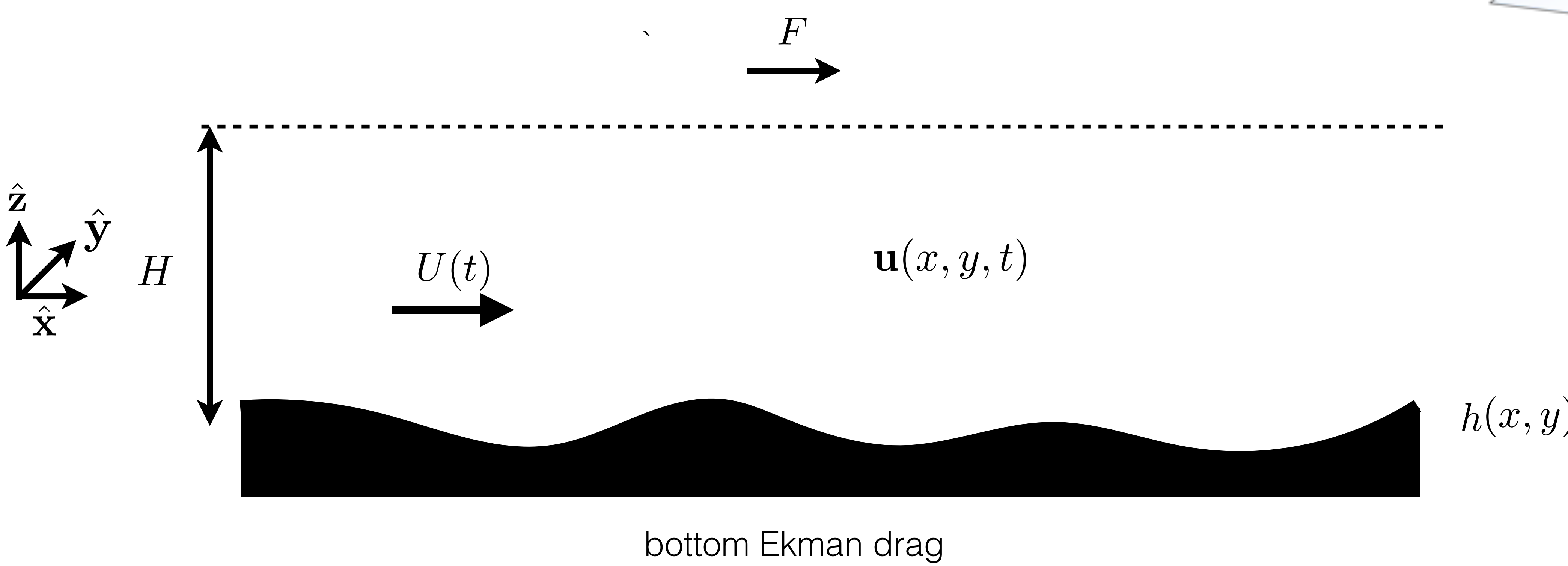
Farneti et. al. 2015

winds seem to be increasing
how will the ACC respond?

doubling the wind gives double the ACC?
not always — “eddy saturation” regime

a single-layer quasi-geostrophic model for the ACC on a beta-plane





dynamical variables

potential vorticity $$q(x, y, t) = f_0 + \beta y + \nabla^2 \psi(x, y, t) + \underbrace{\frac{f_0 h(x, y)}{H}}_{=\eta(x,y)}$$

large-scale zonal flow $$U(t)$$

flow evolution

$$\partial_t q + \mathsf{J}(\psi - Uy, q) = -\mu\nabla^2\psi$$
$$\partial_t U = F - \mu U - \langle \psi \partial_x \eta \rangle$$
$$q = f_0 + \beta y + \nabla^2\psi + \eta$$

Hart 1979
Carnevale & Frederiksen 1989
Holloway 1989

domain = square of length L

periodic boundary
conditions in x,y

$$\langle \ \bullet \ \rangle = \frac{1}{L^2}\int \bullet \, \mathrm{d}^2\mathbf{x}$$

parameters

$$\partial_t q + \mathsf{J}(\psi - Uy, q) = -\mu \nabla^2 \psi$$
$$\partial_t U = F - \mu U - \langle \psi \partial_x \eta \rangle$$
$$q = f_0 + \beta y + \nabla^2 \psi + \eta$$

F : mean wind stress

β : planetary vorticity gradient, $\beta = \mathrm{d}f/\mathrm{d}y\Big|_{\theta=\theta_0}$

μ : bottom Ekman drag coefficient

$\eta(\mathbf{x})$: topography

$\eta_{\mathrm{rms}} = \sqrt{\langle \eta^2 \rangle}$

$\ell_\eta = \sqrt{\eta_{\mathrm{rms}}^2 / \langle |\boldsymbol{\nabla} \eta|^2 \rangle}$

spectral distribution (e.g. isotropic)

spectral slope (for isotropic)

energy & potential enstrophy

$$\partial_t q + \mathsf{J}(\psi - Uy, q) = -\mu\nabla^2\psi$$
$$\partial_t U = F - \mu U - \langle \psi \partial_x \eta \rangle$$
$$q = f_0 + \beta y + \nabla^2\psi + \eta$$

$$E = \underbrace{\frac{1}{2}\langle |\boldsymbol{\nabla}\psi|^2\rangle}_{E_\psi} + \underbrace{\frac{1}{2}U^2}_{E_U}$$ energy

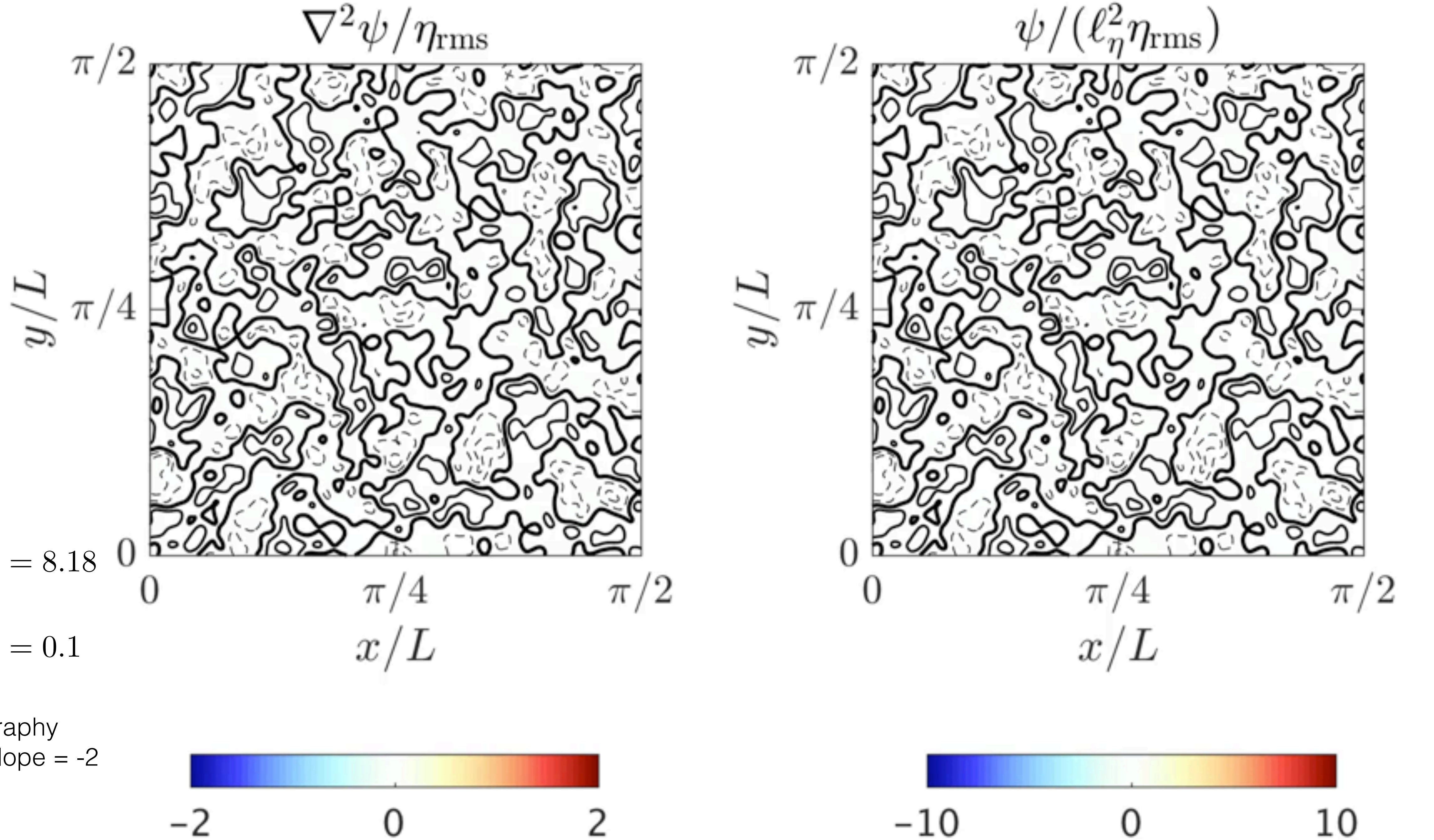
$$Q = \underbrace{\frac{1}{2}\langle (\nabla^2\psi + \eta)^2\rangle}_{Q_\psi} + \underbrace{\beta U}_{Q_U}$$ potential enstrophy

$$\frac{\mathrm{d}E}{\mathrm{d}t} = FU - \mu U^2 - \mu\langle |\boldsymbol{\nabla}\psi|^2\rangle$$
$$\frac{\mathrm{d}Q}{\mathrm{d}t} = F\beta - \mu\beta U - \mu\langle (\nabla^2\psi + \eta)\nabla^2\psi\rangle$$

total energy and potential enstrophy are conserved
in the absence of forcing and dissipation

topography

$$\mu t = 0.00$$

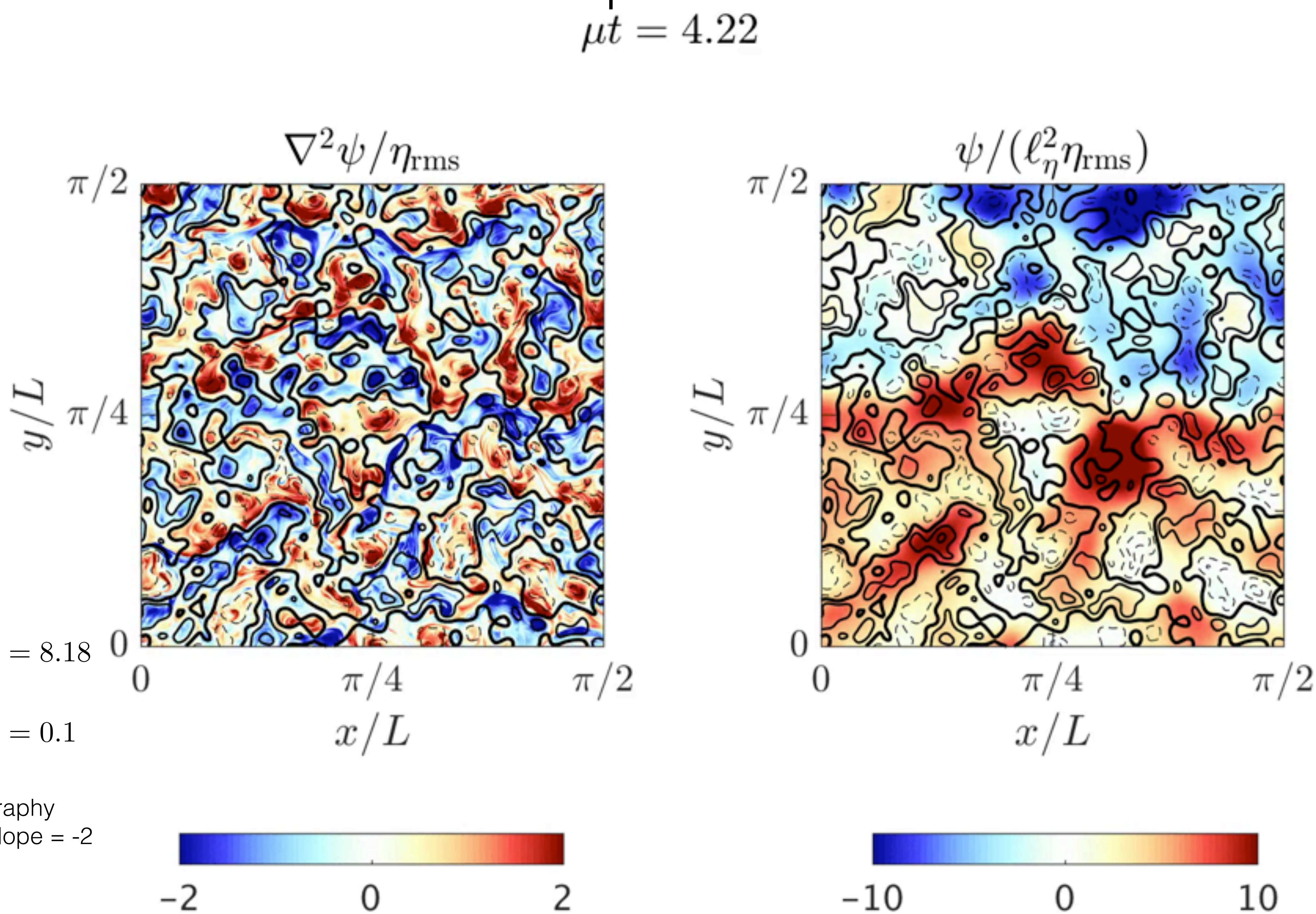


$$\frac{F}{\mu\eta_{\mathrm{rms}}\ell_\eta} = 8.18$$

$$\frac{\beta\ell_\eta}{\eta_{\mathrm{rms}}} = 0.1$$

topography
spectral slope = -2

a snapshot of the flow at statistically steady state for "realistic" parameter values



$$\frac{F}{\mu\eta_{\mathrm{rms}}\ell_\eta} = 8.18$$

$$\frac{\beta\ell_\eta}{\eta_{\mathrm{rms}}} = 0.1$$

topography
spectral slope = -2

topographic form stress

$$\overline{\cancel{\partial_t U}} = \overline{F - \mu U - \langle \psi \partial_x \eta \rangle}$$

$=0$

topographic form stress
(or pressure drag)
(or mountain drag)
(or form drag)

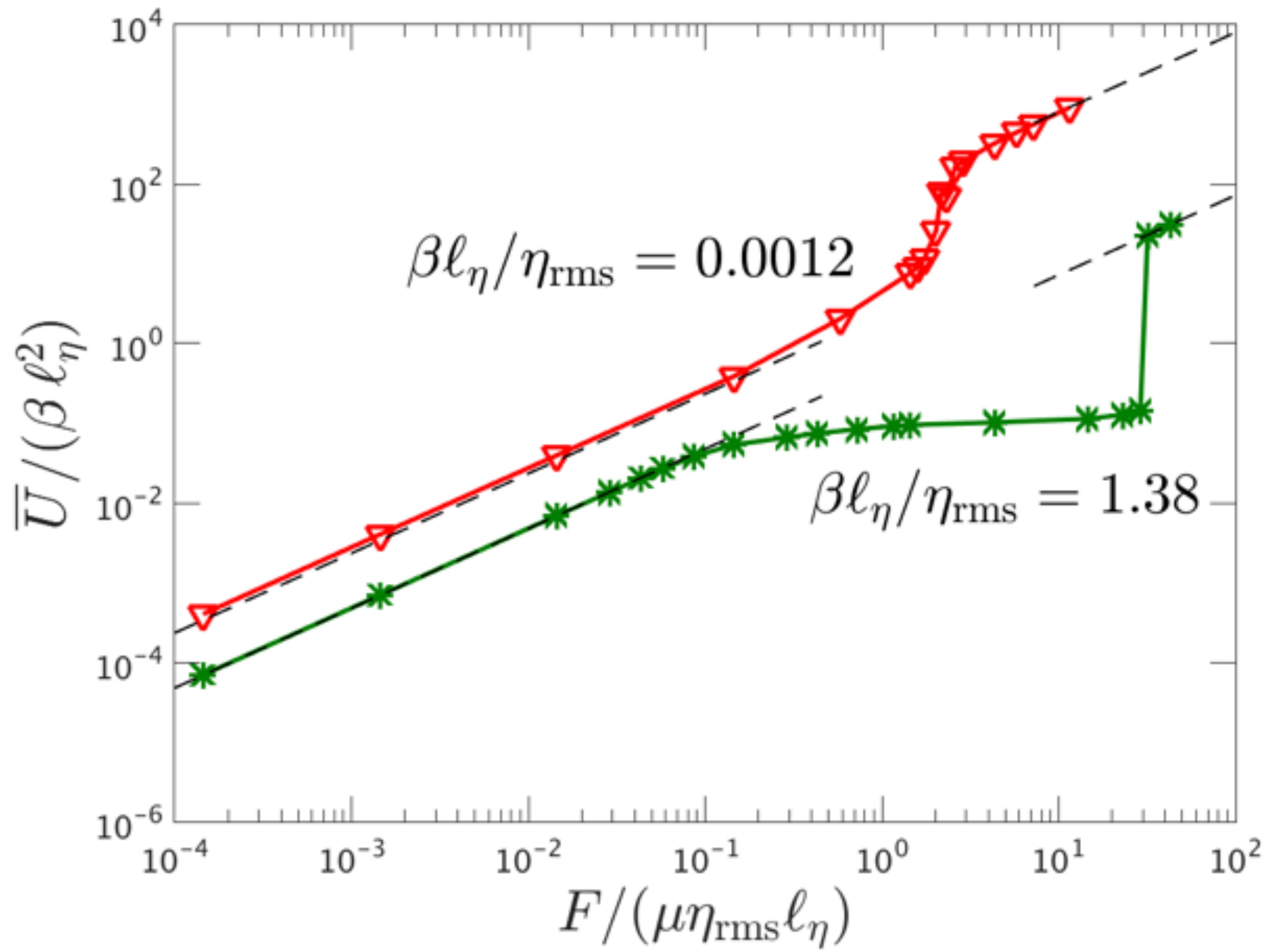
form stress controls the steady state large-scale *U*

for a flat bottom $\overline{U} = \dfrac{F}{\mu}$ very large (Munk & Palmen 1951)

for a non-flat bottom $\overline{U} = \dfrac{F - \overline{\langle \psi \partial_x \eta \rangle}}{\mu}$

some numerical results…

fix everything and vary the wind stress



some numerical results…

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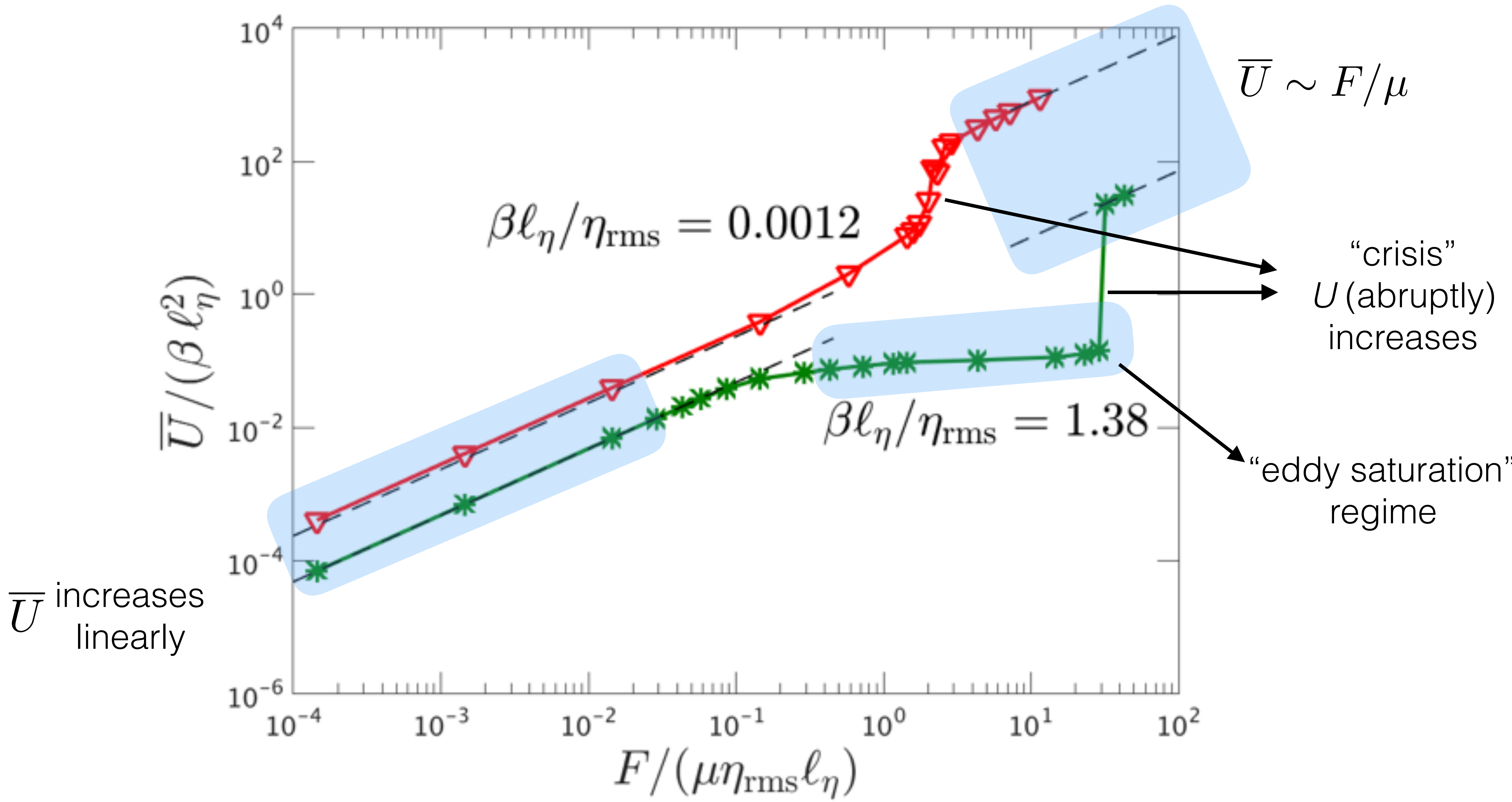


a bound for the form stress based on the energy equation

let's try maximize $\langle\psi\partial_x\eta\rangle$ given the constraints:
mean flow + energy are at stationary steady state

a bound for the form stress
based on the energy equation

let's try maximize $\langle \psi \partial_x \eta \rangle$ given the constraints:
mean flow + energy are at stationary steady state

Note that: $$\frac{\mathrm{d}E_U}{\mathrm{d}t} = \frac{U}{\beta}\frac{\mathrm{d}Q_U}{\mathrm{d}t} = U\frac{\mathrm{d}U}{\mathrm{d}t}$$

therefore

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \Leftrightarrow \frac{\mathrm{d}E_\psi}{\mathrm{d}t} = 0 \ \& \ \frac{\mathrm{d}U}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0 \Leftrightarrow \frac{\mathrm{d}Q_\psi}{\mathrm{d}t} = 0 \ \& \ \frac{\mathrm{d}U}{\mathrm{d}t} = 0$$

a bound for the form stress based on the energy equation

$$\mathcal{F}\left[\psi\right] = \overline{\langle\psi\partial_x\eta\rangle} + \lambda_1\left(F - \mu\overline{U} - \overline{\langle\psi\partial_x\eta\rangle}\right) + \lambda_2\left(\overline{U\langle\psi\partial_x\eta\rangle} - \mu\overline{\langle|\boldsymbol{\nabla}\psi|^2\rangle}\right)$$

steady state
mean flow
equation

steady state
energy eq.
for ψ

$$\boxed{\overline{\langle\psi\partial_x\eta\rangle} \leq \frac{F}{1+\mu^2\left(\sum_{\mathbf{k}}\frac{k_x^2}{|\mathbf{k}|^2}|\hat{\eta}(\mathbf{k})|^2\right)^{-1}}}$$

proportional to F!

a bound for the form stress based on the enstrophy equation

$$\mathcal{F}\left[\psi\right] = \overline{\langle\psi\partial_x\eta\rangle} + \lambda_1\left(F - \mu\overline{U} - \overline{\langle\psi\partial_x\eta\rangle}\right) + \lambda_2\left(\beta\overline{\langle\psi\partial_x\eta\rangle} - \mu\overline{\langle(\nabla^2\psi+\eta)\nabla^2\psi\rangle}\right)$$

steady state mean flow equation

steady state enstrophy eq. for ψ

$$\boxed{\overline{\langle\psi\partial_x\eta\rangle} \le \frac{(\sqrt{\beta^2+\kappa}+\beta)}{2\mu}\sum_{\mathbf{k}}\frac{k_x^2}{|\mathbf{k}|^4}|\hat{\eta}(\mathbf{k})|^2}$$

$$\kappa = \frac{\mu^2\eta_{\mathrm{rms}}^2}{\sum_{\mathbf{k}}\frac{k_x^2}{|\mathbf{k}|^4}|\hat{\eta}(\mathbf{k})|^2}$$

independent of F!

a bound for the form stress based on the energy + enstrophy equation

$$\mathcal{F}\left[\psi\right] = \overline{\langle \psi \partial_x \eta \rangle} + \lambda_1 \left(F - \mu \overline{U} - \overline{\langle \psi \partial_x \eta \rangle} \right) + \lambda_2 \left(\overline{U \langle \psi \partial_x \eta \rangle} - \mu \overline{\langle |\boldsymbol{\nabla} \psi|^2 \rangle} \right)$$
$$+ \lambda_3 \left(\beta \overline{\langle \psi \partial_x \eta \rangle} - \mu \overline{\langle (\nabla^2 \psi + \eta) \nabla^2 \psi \rangle} \right)$$

do the algebra…

a bound for the form stress based on the energy + enstrophy equation

$$\overline{\langle \psi \partial_x \eta \rangle} \leq \min \left\{ \frac{F}{1 + \mu^2 \left(\sum_{\mathbf{k}} \frac{k_x^2}{|\mathbf{k}|^2} |\hat{\eta}(\mathbf{k})|^2 \right)^{-1}}, \frac{(\sqrt{\beta^2 + \kappa} + \beta)}{2\mu} \sum_{\mathbf{k}} \frac{k_x^2}{|\mathbf{k}|^4} |\hat{\eta}(\mathbf{k})|^2 \right\}$$

essentially no new information… :(

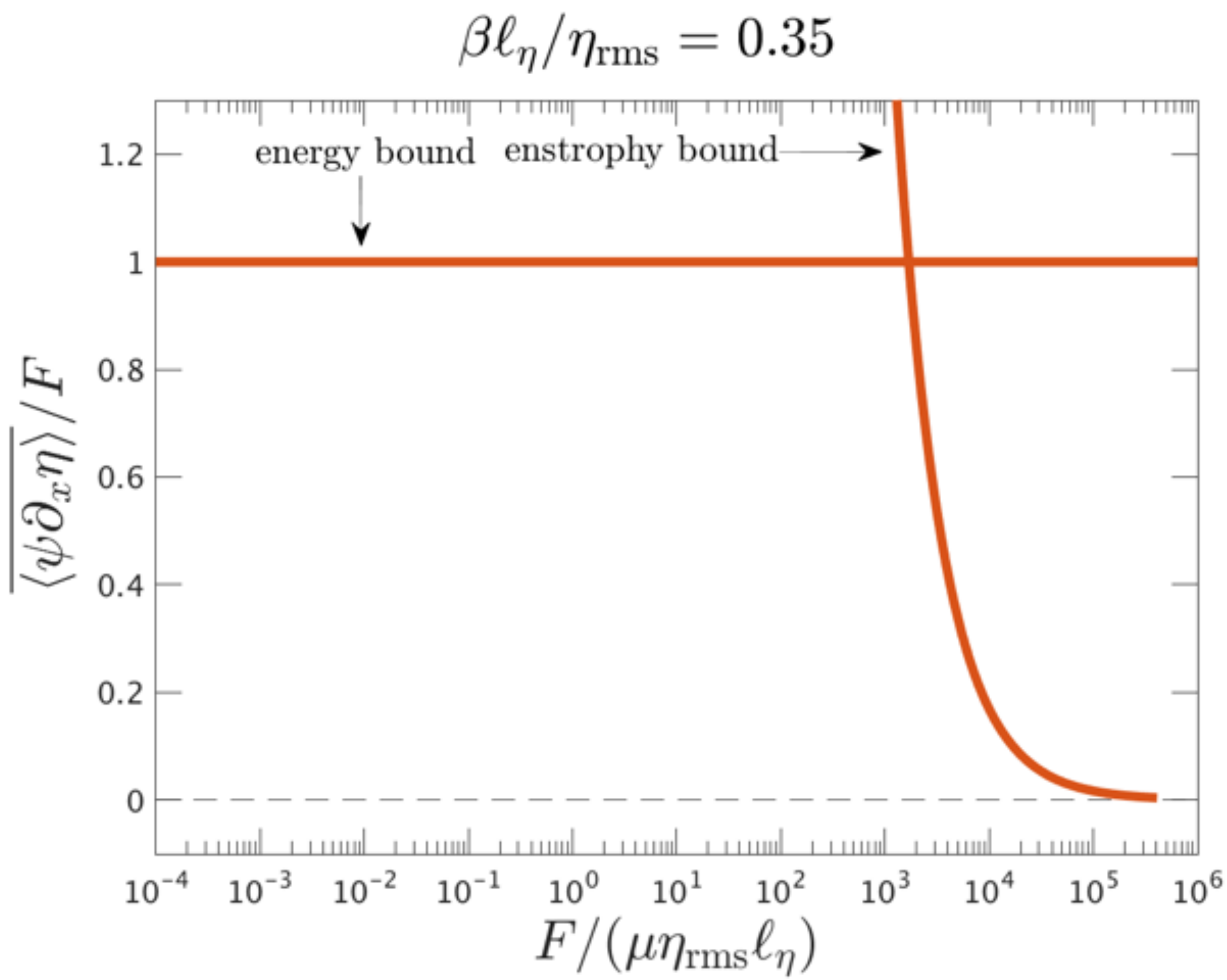
what do these bounds imply?

$$\underbrace{\cancel{\overline{\partial_t U}}}_{=0} = \overline{F - \mu U - \langle \psi \partial_x \eta \rangle}$$

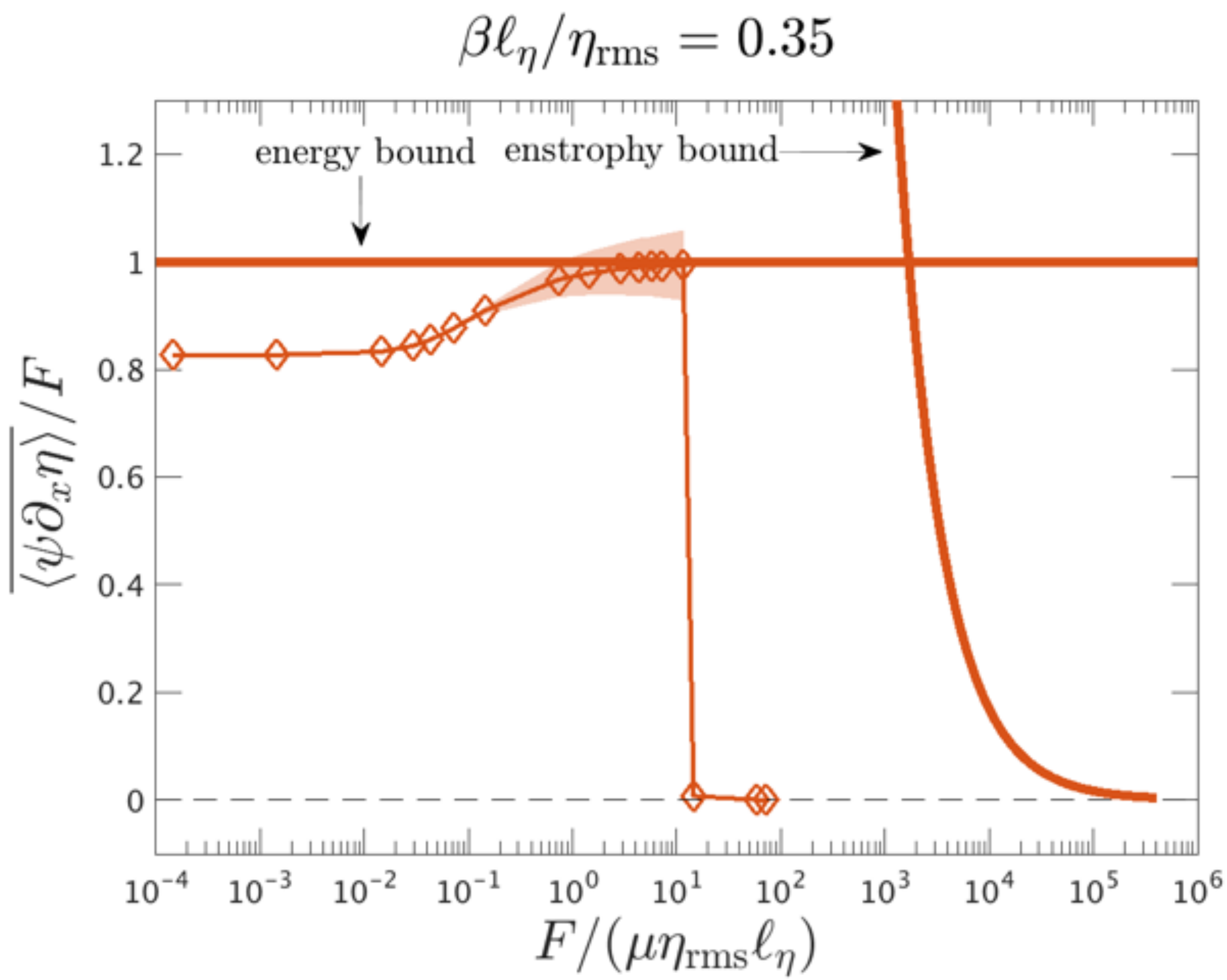
$$1 = \frac{\mu \overline{U}}{F} + \frac{\overline{\langle \psi \partial_x \eta \rangle}}{F}$$

the momentum imparted by *F*
is balanced by who?

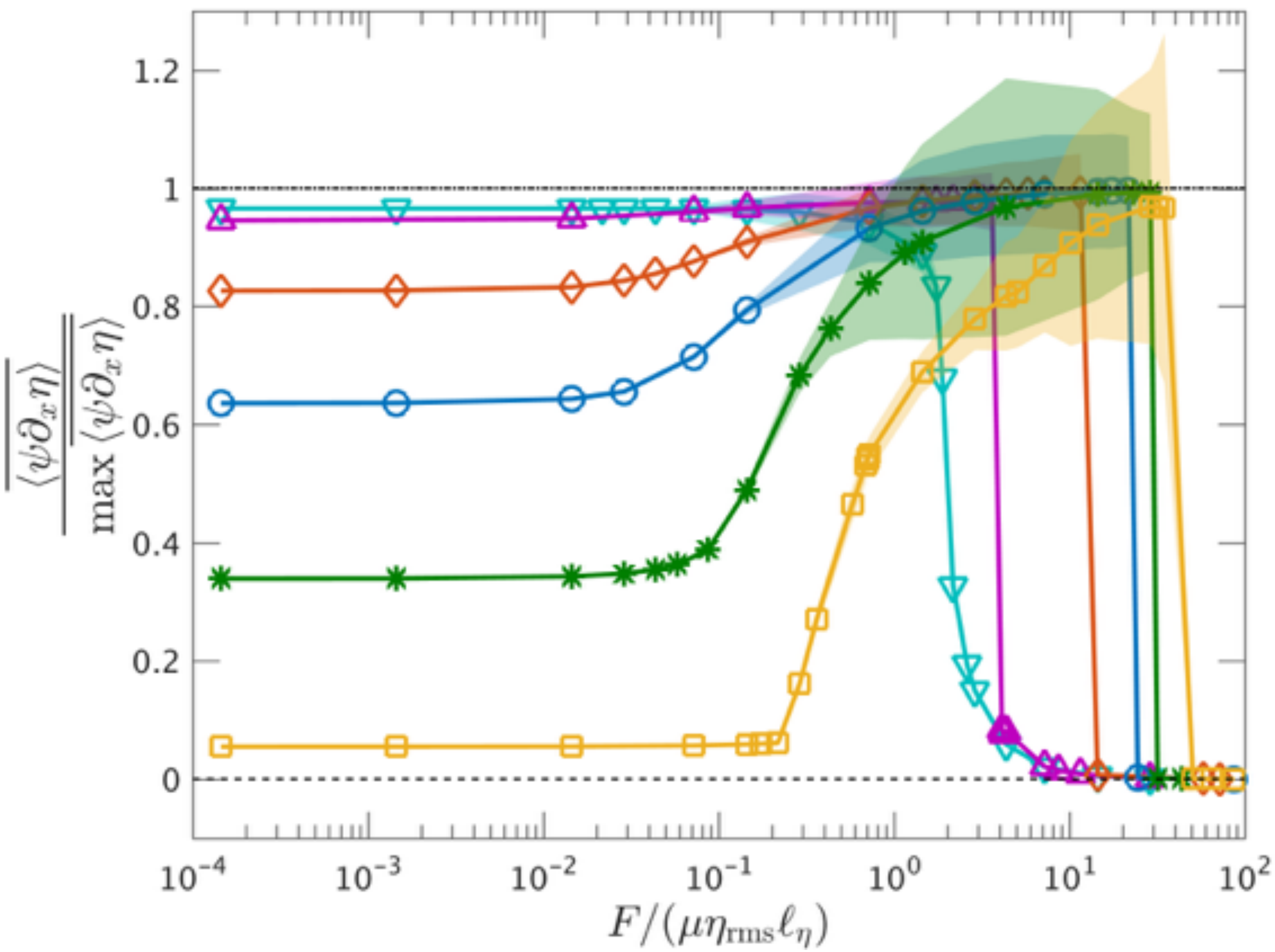
what do these bounds imply?



numerical resuts



how does the form stress respond to wind increase?



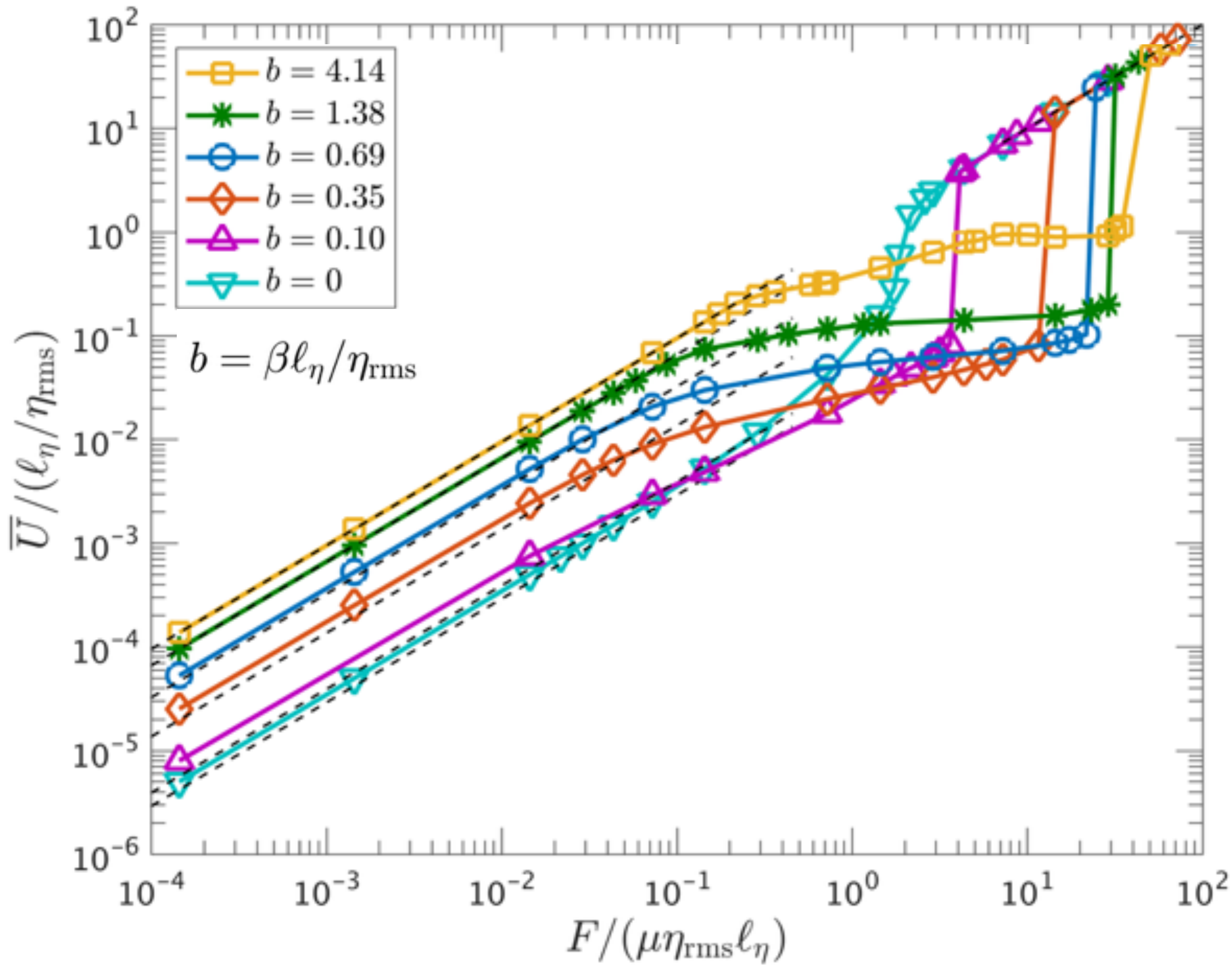
form stress picks up
and
suddenly we have a “crisis”:
form stress disappears and
all momentum is balanced by *U*

→ *U*=large

$b = \beta \ell_\eta / \eta_{\text{rms}}$

“crisis” occurs for *b>0.01*

how does U respond to wind increase?



the regime $\dfrac{F}{\mu \eta_{\mathrm{rms}} \ell_\eta} \ll 1$ & $b = \beta \ell_\eta / \eta_{\mathrm{rms}} \gtrapprox O(1)$

assuming a regular perturbation expansion for ψ and U
we get that to first order

$$\mathsf{J}(\psi - Uy, \eta + \beta y) = 0$$
$$F - \mu U - \langle \psi \partial_x \eta \rangle = 0$$

and using the eddy energy equation

$$U_0 = \frac{F/\mu}{1 + 1/b^2} \quad , \quad \langle \psi_0 \partial_x \eta \rangle = \frac{F}{1 + b^2}$$

the regime $\frac{F}{\mu\eta_{\mathrm{rms}}\ell_\eta} \ll 1$ & $b = \beta\ell_\eta/\eta_{\mathrm{rms}}$ =0

it turns out that the problem
is mathematically homomorphic to the steady state
solution of the advection of a passive scalar by a flow
in the presence of a large-scale concentration gradient

the regime $\frac{F}{\mu \eta_{\mathrm{rms}} \ell_\eta} \ll 1$ & $b = \beta \ell_\eta / \eta_{\mathrm{rms}}$ =0

it turns out that the problem
is mathematically homomorphic to the steady state
solution of the advection of a passive scalar by a flow
in the presence of a large-scale concentration gradient

$$\mathsf{J}(\phi, c - Gy) = \kappa \nabla^2 c \qquad \mathsf{J}(\eta, \psi - Uy) = \mu \nabla^2 \psi$$

streamfunction

large-scale
concentration
gradient

passive
scalar
concentration

the analogy

$\mathsf{J}(\phi, c - Gy) = \kappa\nabla^2 c$	$\mathsf{J}(\eta, \psi - Uy) = \mu\nabla^2\psi$
streamfunction	topography
concentration	streamfunction
diffusion coefficient	dissipation coefficient
large-scale conc. gradient	large-scale flow

$$\mathrm{Pe} = \phi_{\mathrm{rms}}/\kappa \qquad \mathrm{Pe}_\eta = \eta_{\mathrm{rms}}/\mu$$

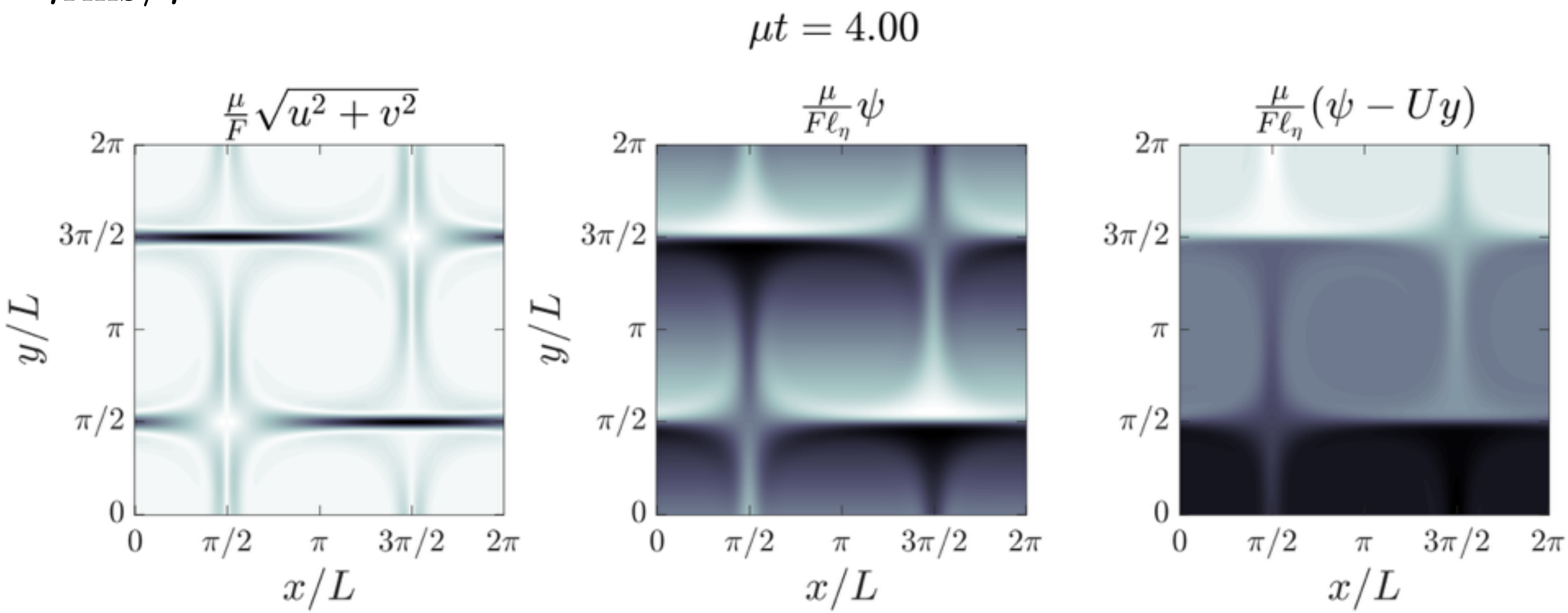
$$\mathrm{Nu} = 1 + \frac{\langle c\partial_x\phi\rangle}{\kappa G} \qquad \mathrm{Nu}_\eta = 1 + \frac{\langle \psi\partial_x\eta\rangle}{\mu U}$$

for cellular flows and high Peclet numbers
the concentration is confined to the places ϕ=0

?

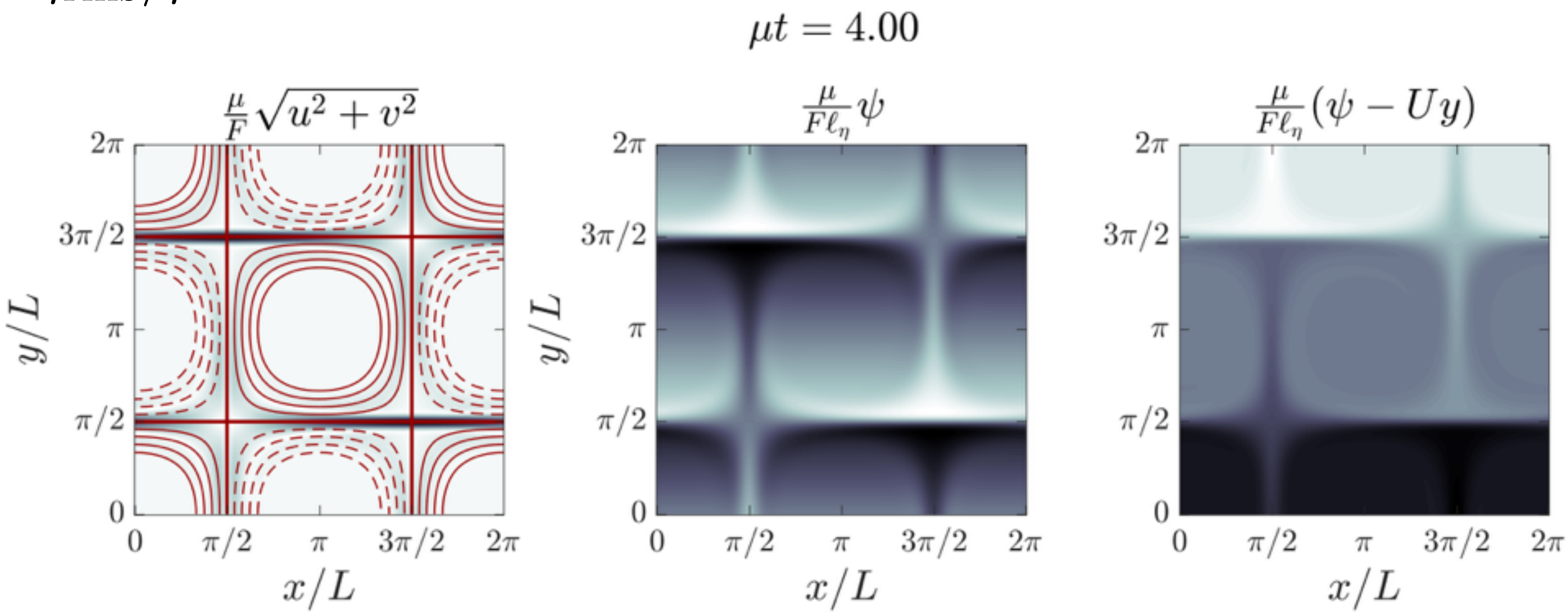
"cellular" topography

$\eta_{\mathrm{rms}}/\mu = 100$



“cellular” topography

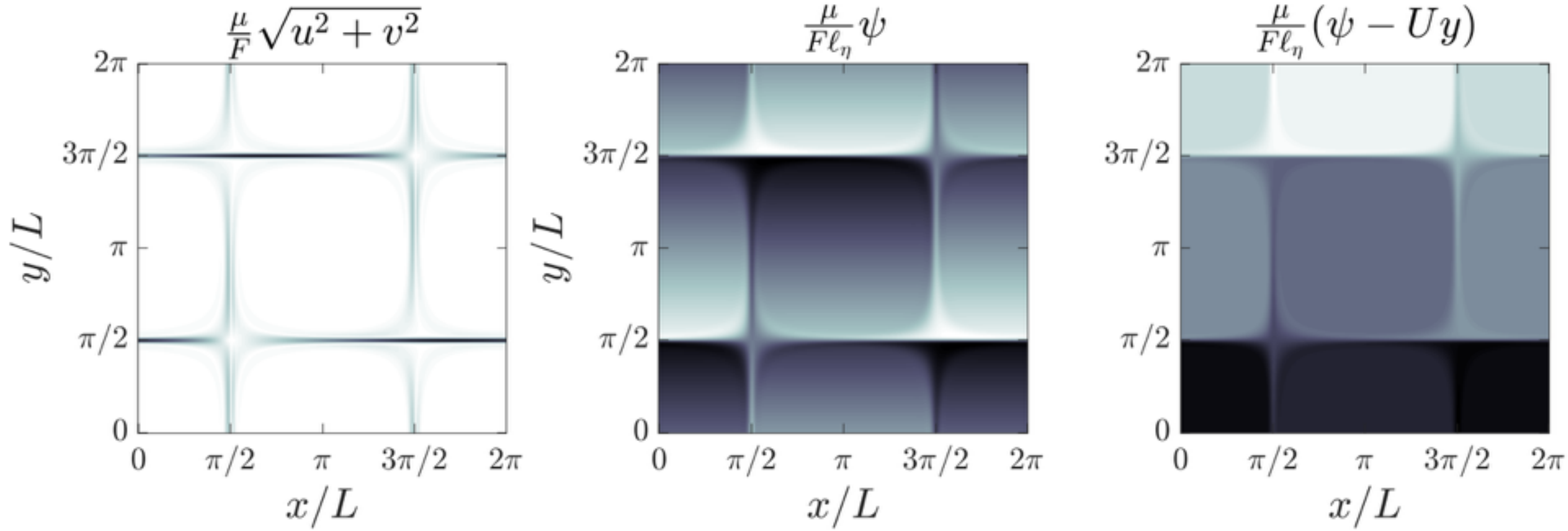
$\eta_{\rm rms}/\mu = 100$



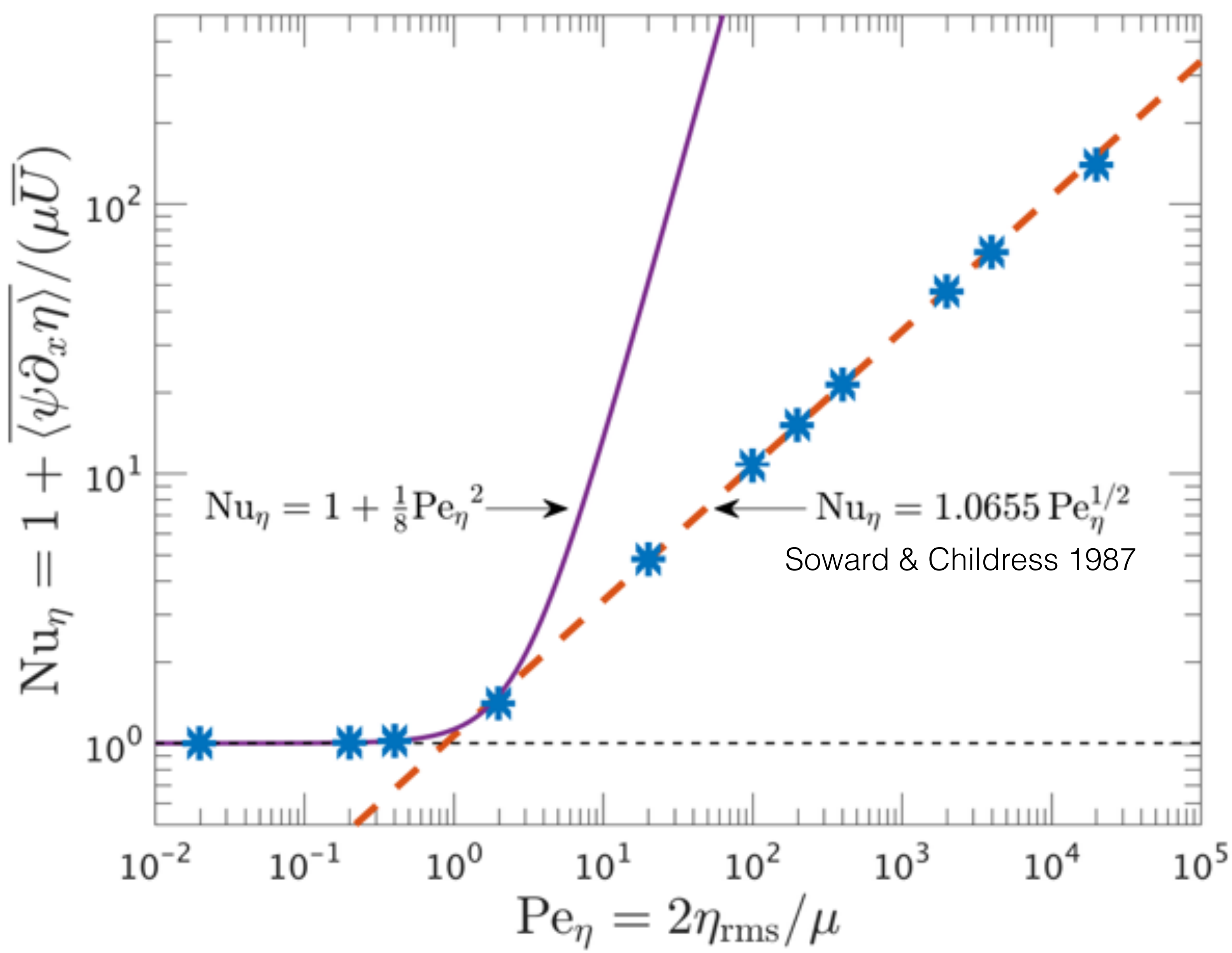
“cellular” topography

$\eta_{\rm rms}/\mu = 1000$

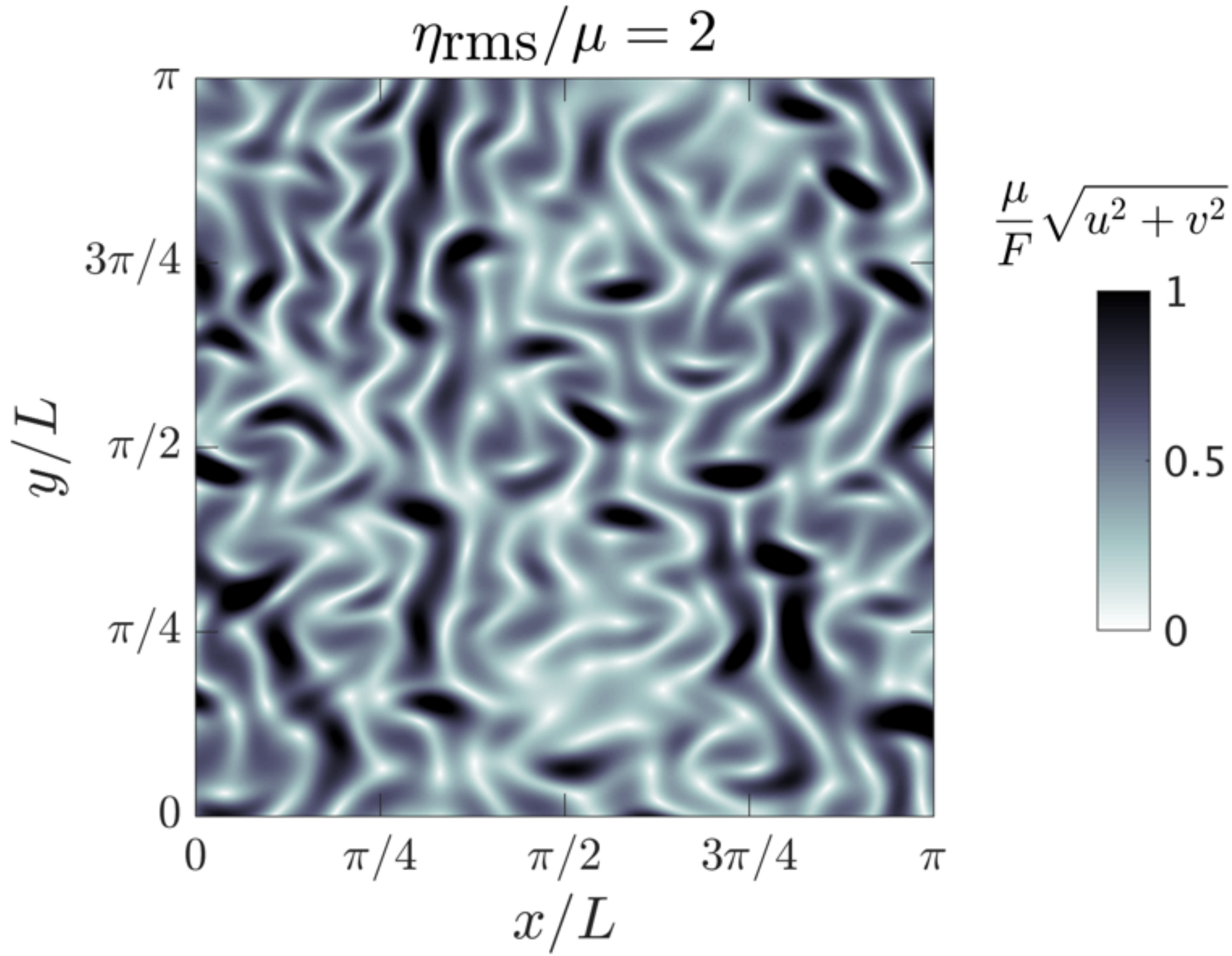
$\mu t = 4.00$



“Nusselt” scaling for “cellular” topography



random monoscale topography



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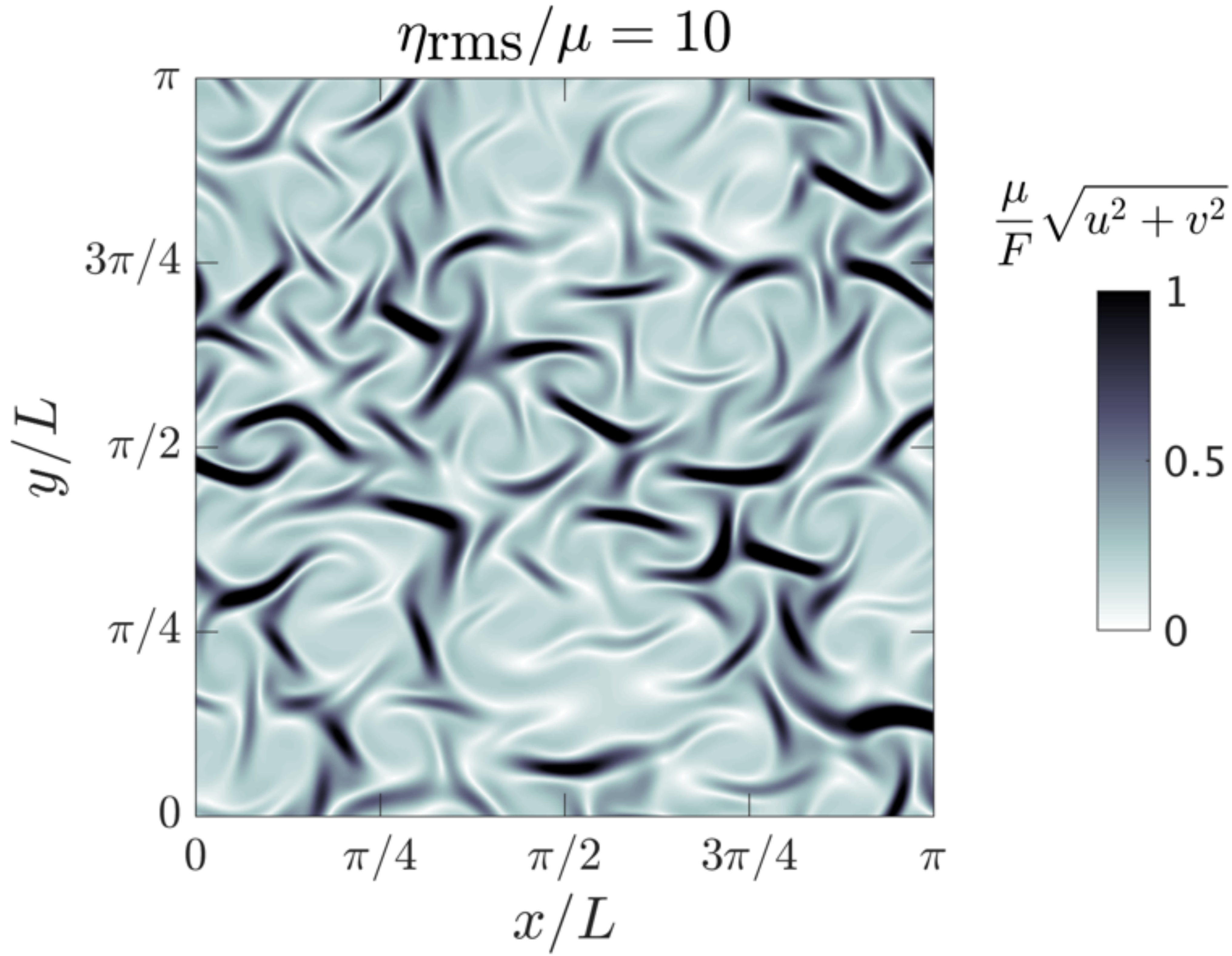

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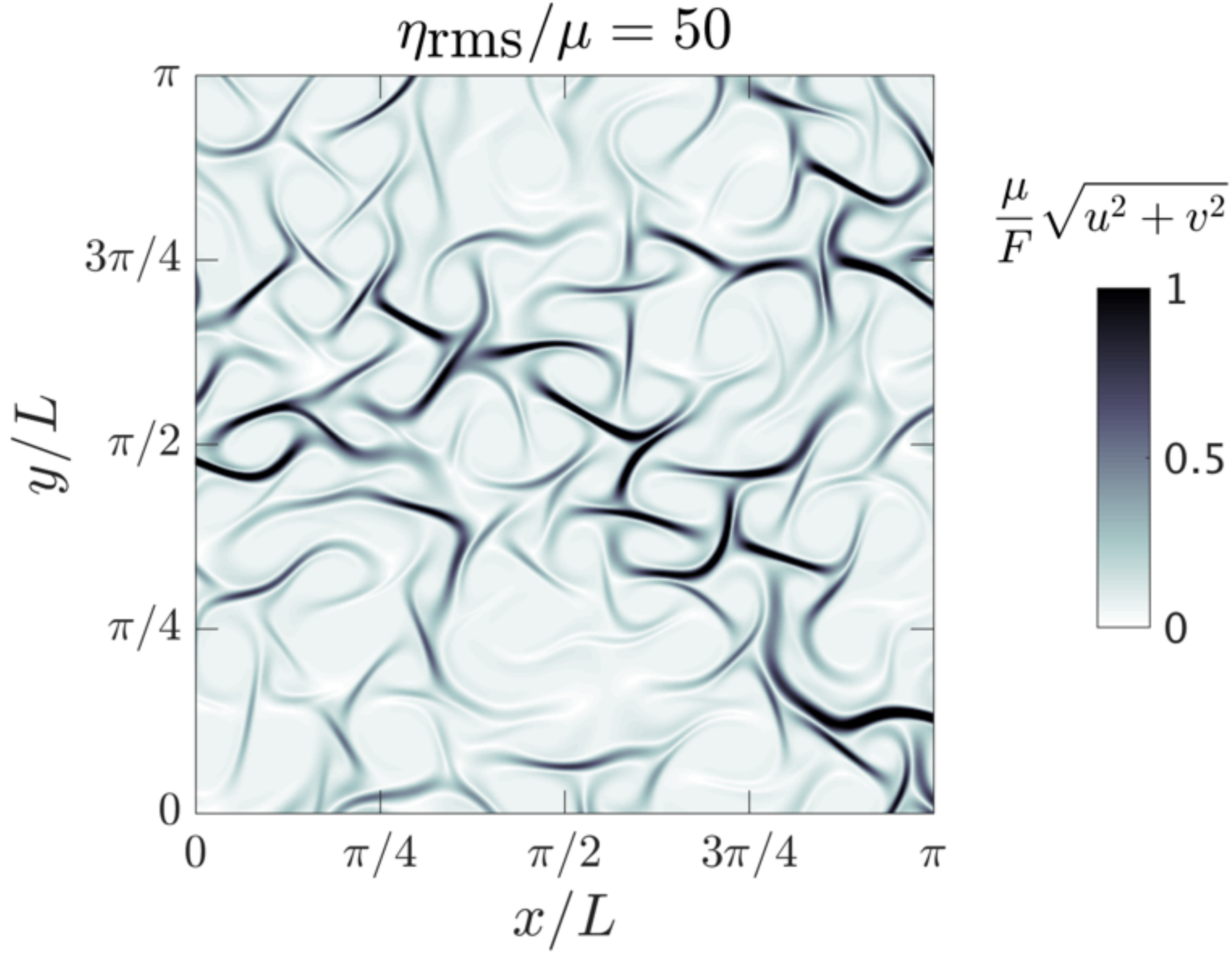

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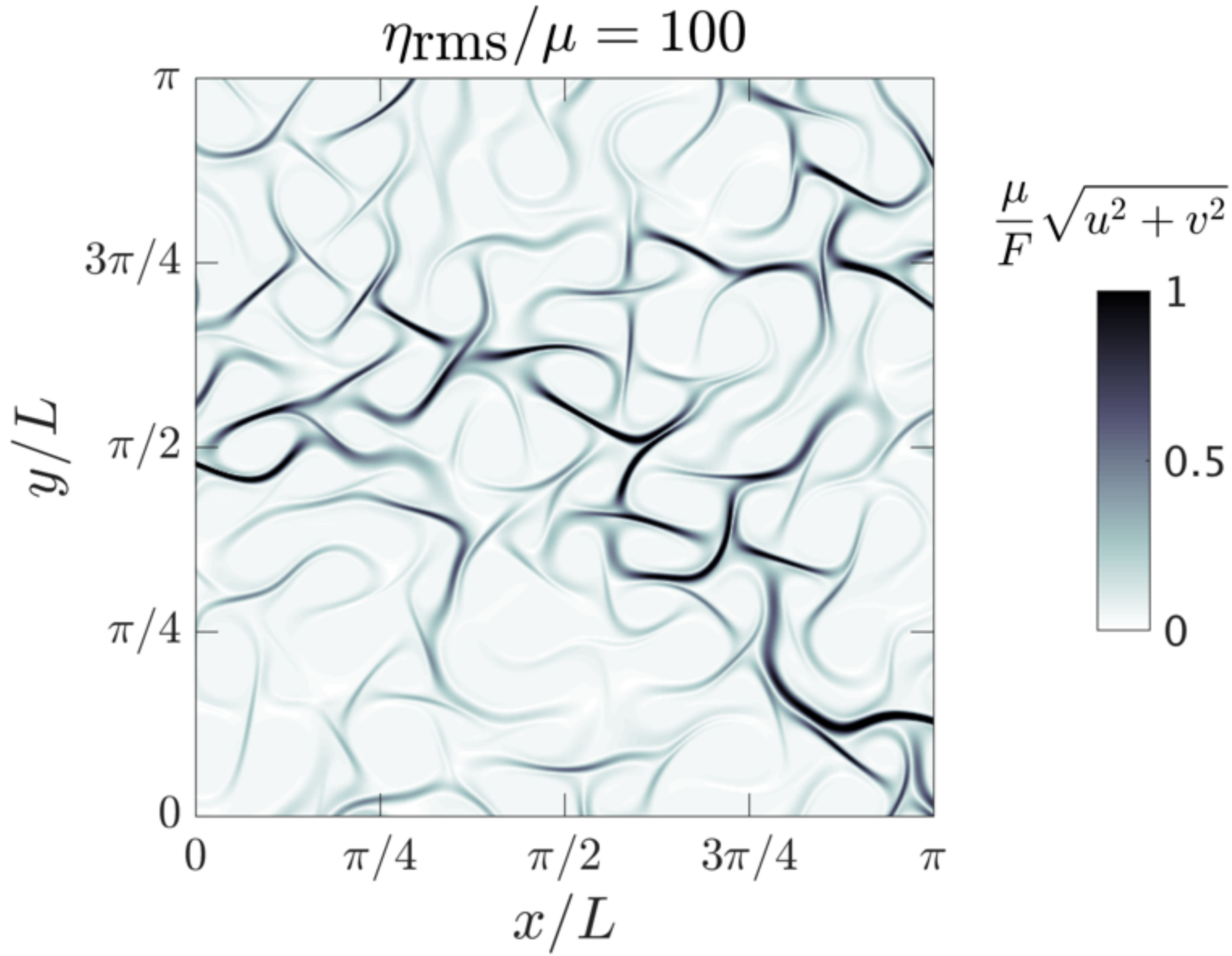

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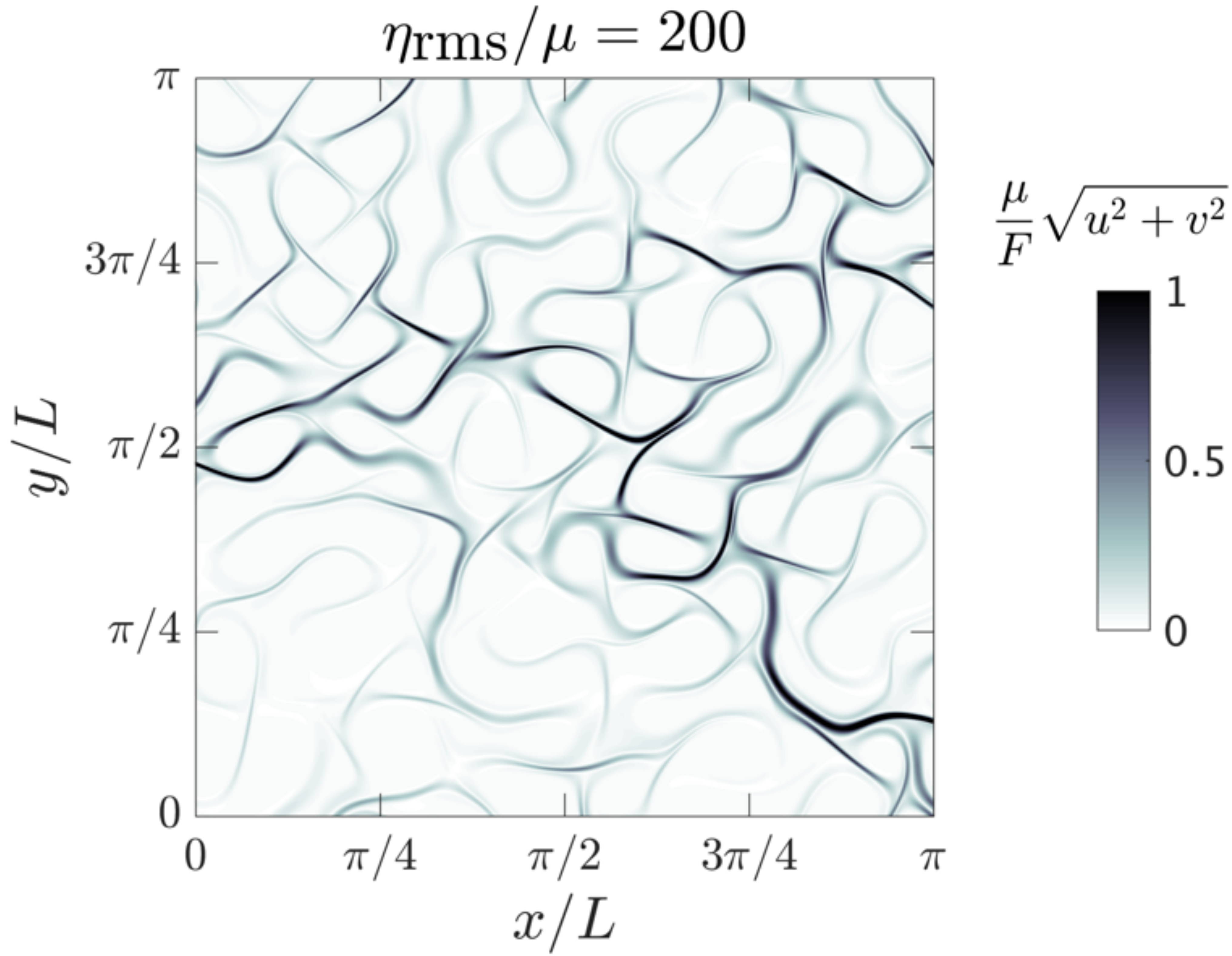

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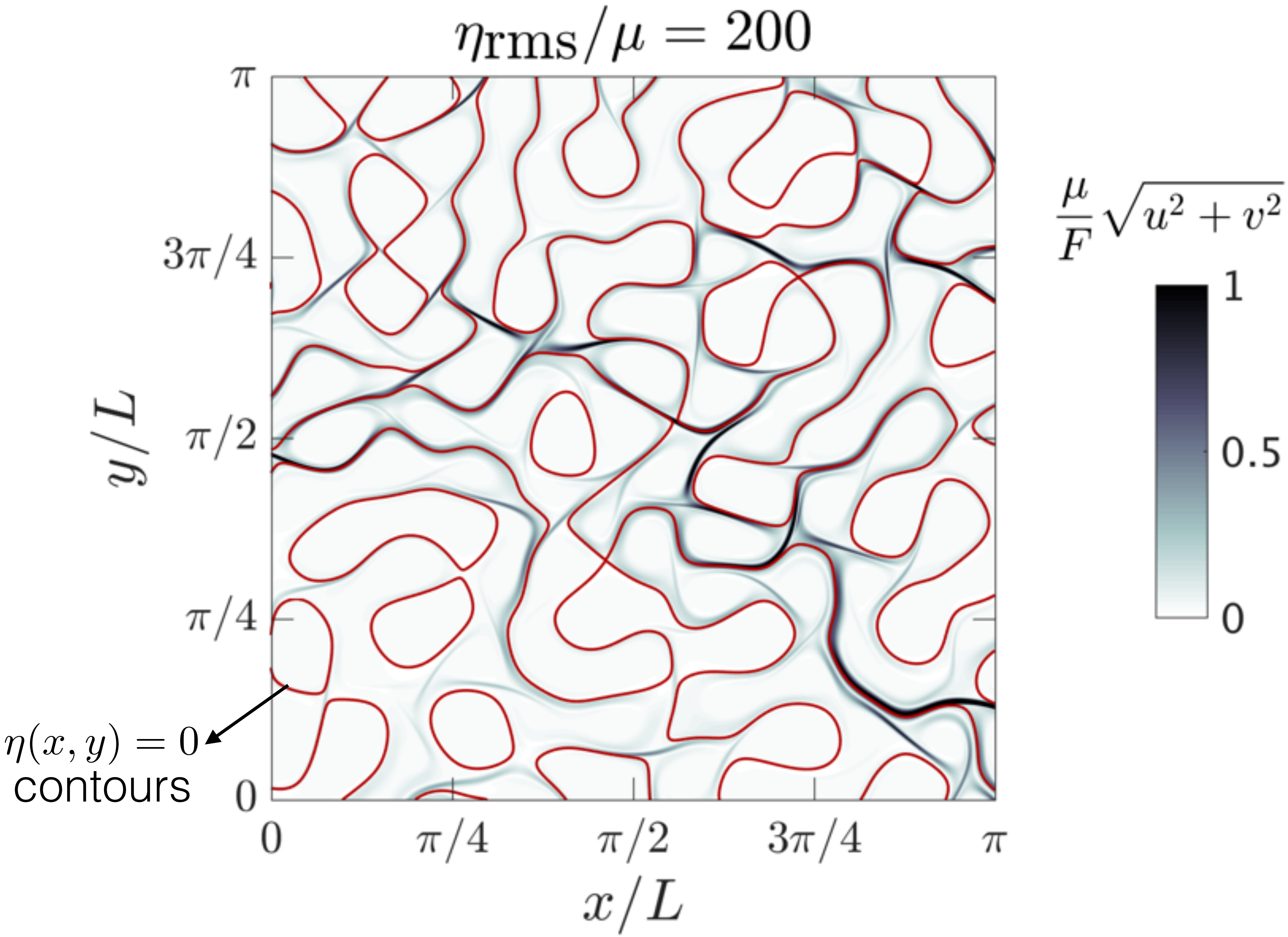
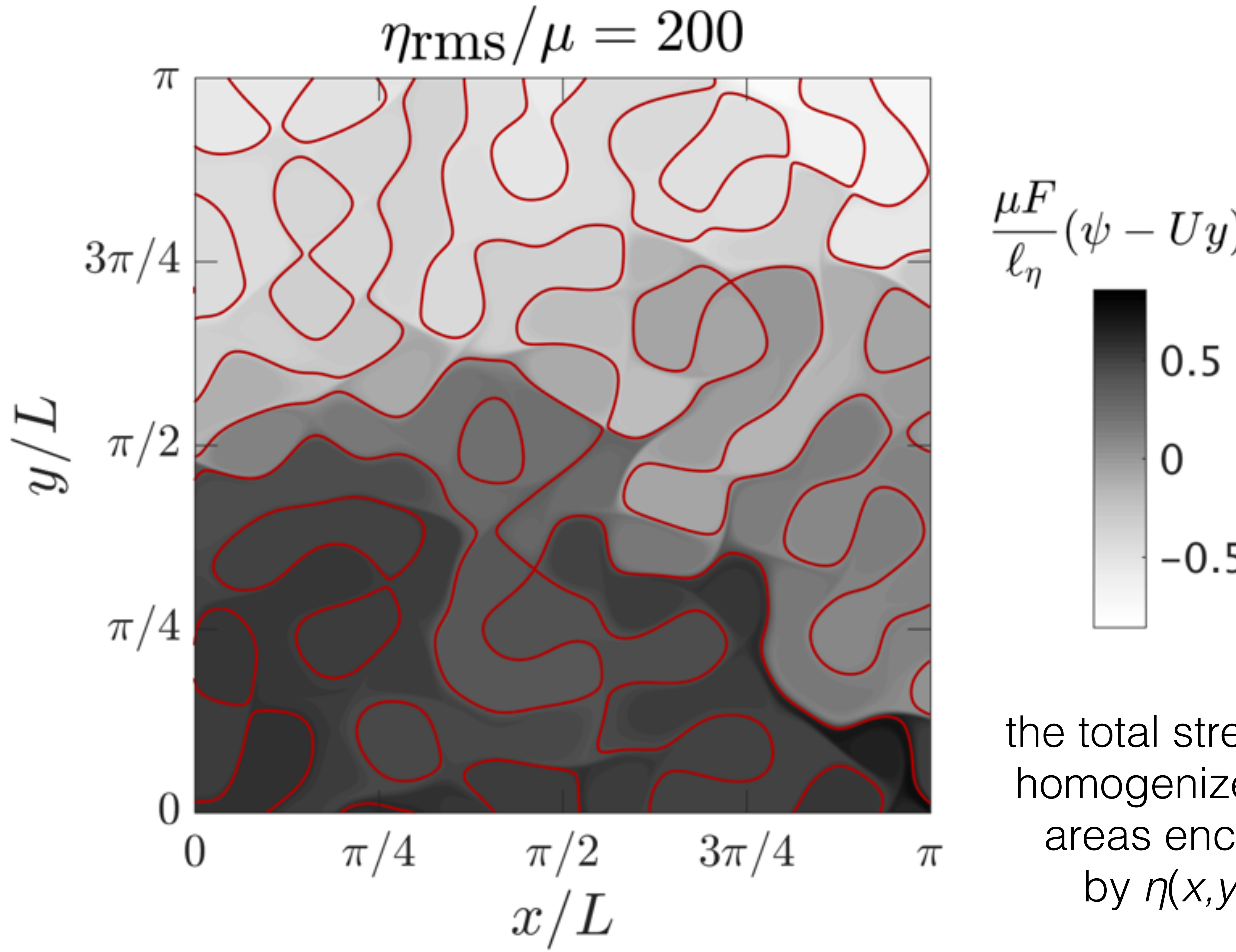
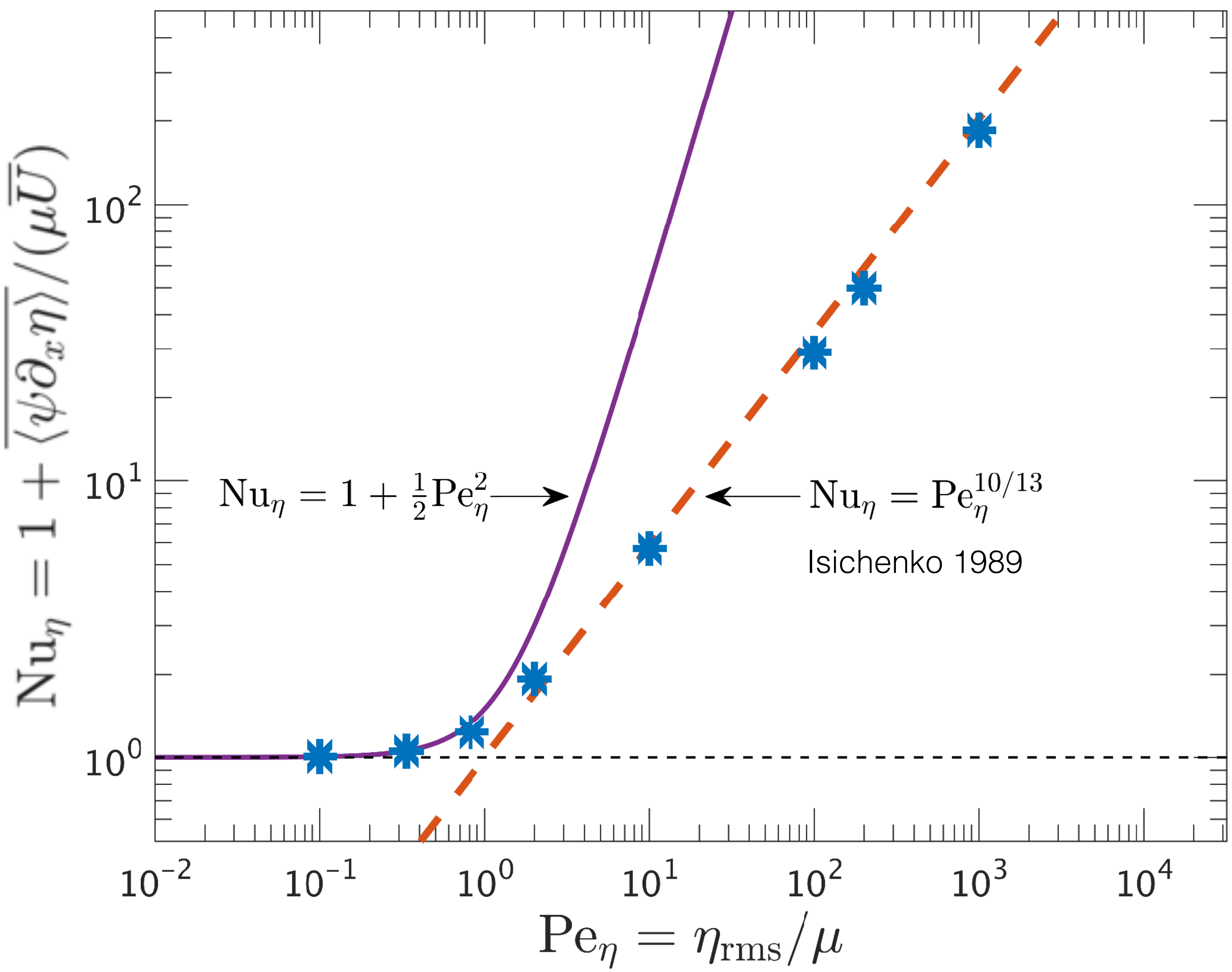


random monoscale topography



the total streamflow homogenizes over areas enclosed by $\eta(x,y)$=0

"Nusselt" scaling
for random monoscale topography



$$\frac{F}{\mu \eta_{\mathrm{rms}} \ell_\eta} \ll 1 \longrightarrow U_0 = \frac{F}{\mu^{3/13} \eta_{\mathrm{rms}}^{10/13}} \quad , \quad \langle \psi_0 \partial_x \eta \rangle = F \left[1 - \left(\frac{\mu}{\eta_{\mathrm{rms}}} \right)^{10/13} \right]$$

b=0

the regime $\dfrac{F}{\mu \eta_{\mathrm{rms}} \ell_\eta} \gg 1$

assuming a regular perturbation expansion for ψ and U
we get to first order:

$$\mathsf{J}(\psi - Uy, \eta + \beta y) = -\mu \nabla^2 \psi$$

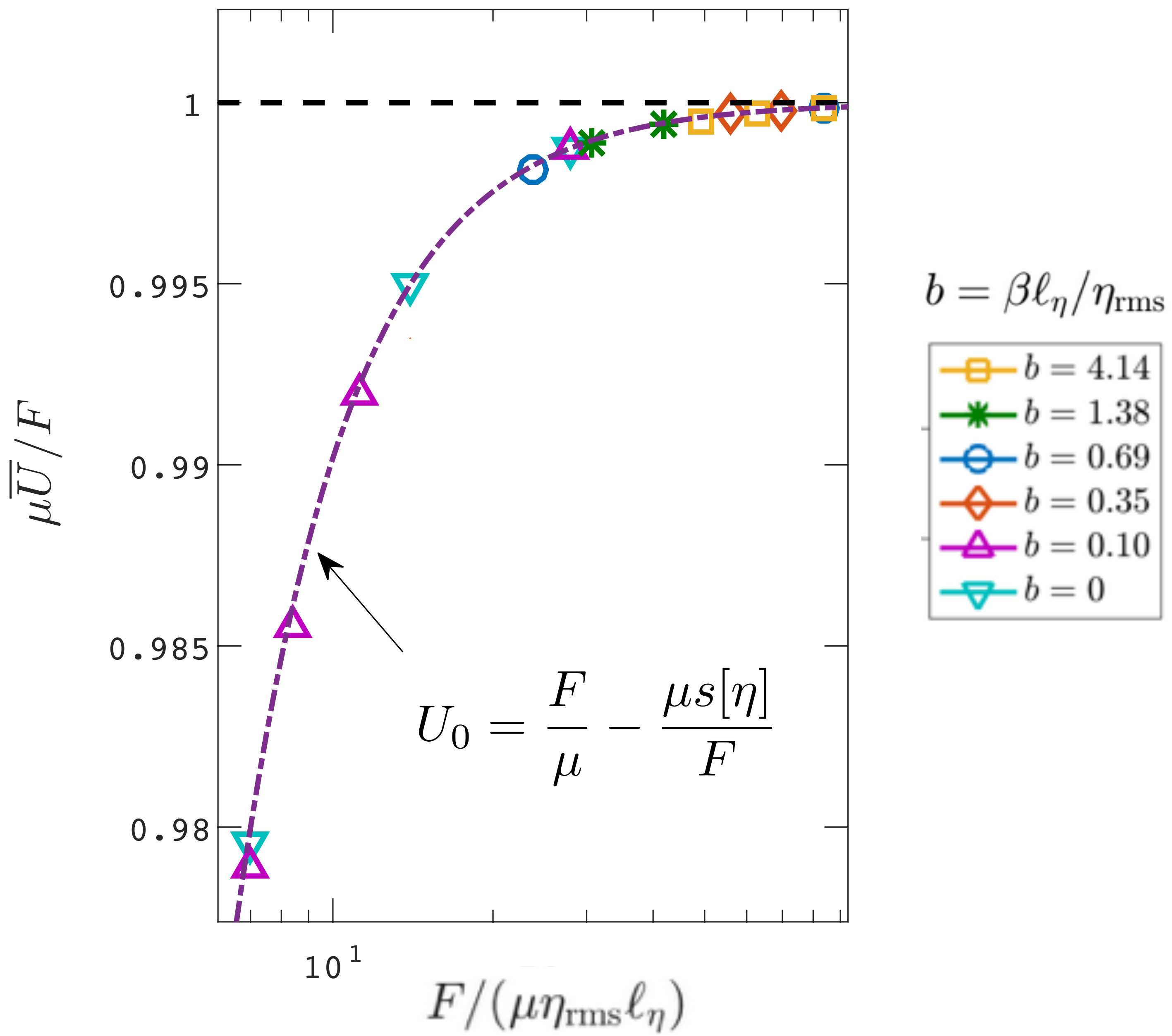
$$F - \mu U - \langle \psi \partial_x \eta \rangle = 0$$

and using the eddy energy equation

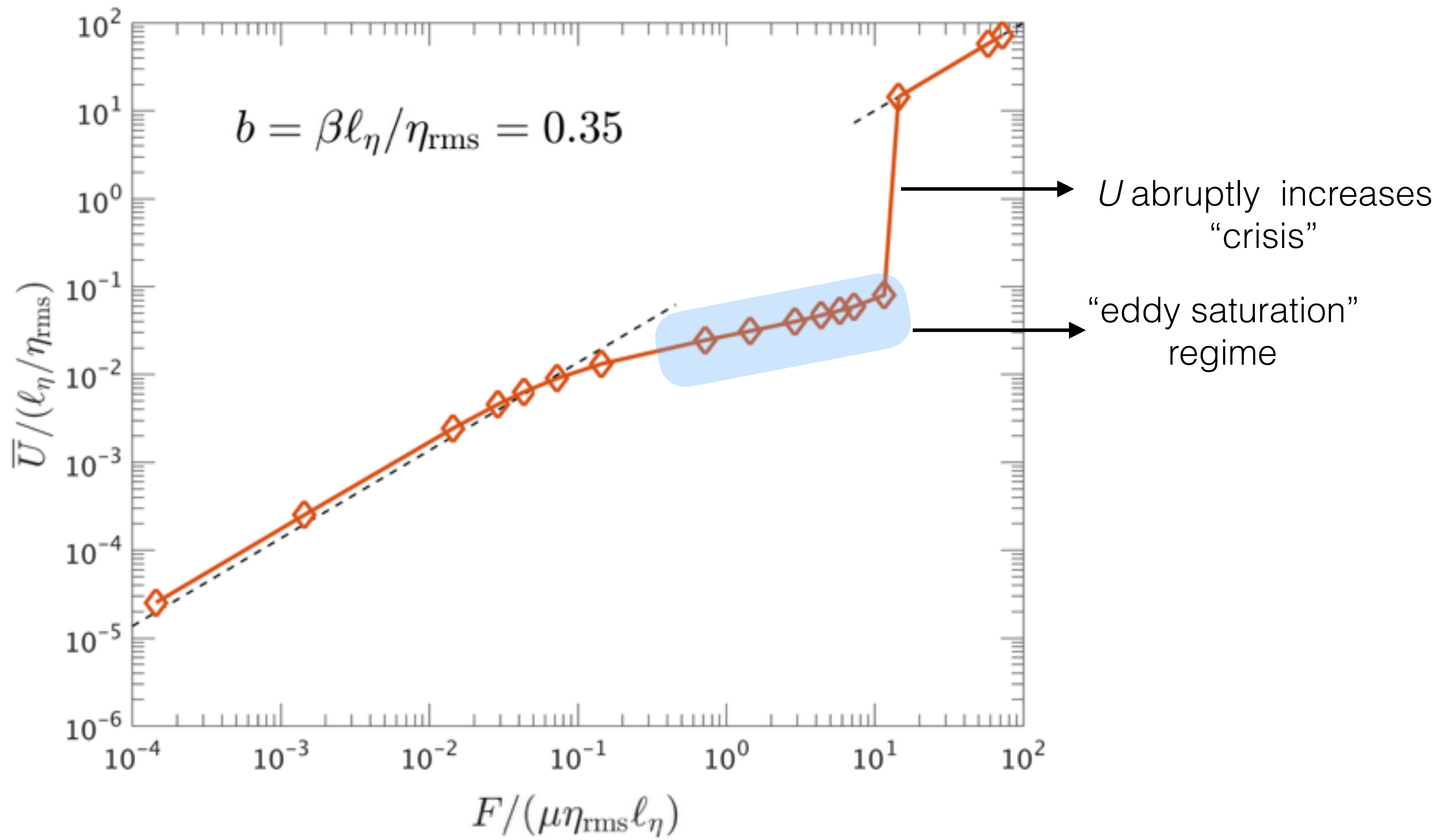
$$U_0 = \frac{F}{\mu} - \frac{\mu s[\eta]}{F} \quad , \quad \langle \psi_0 \partial_x \eta \rangle = \frac{\mu^2 s[\eta]}{F} \qquad s[\eta] = \sum_{\mathbf{k}} \frac{|\hat{\eta}(\mathbf{k})|^2}{|\mathbf{k}|^2}$$

independent of b

the regime $\dfrac{F}{\mu \eta_{\mathrm{rms}} \ell_\eta} \gg 1$



the “eddy saturation” regime & crisis



bound on enstrophy dissipation rate?

if $\nu\langle|\boldsymbol{\nabla}\nabla^2\psi|^2\rangle \lesssim F^p$, *p*<1 this will imply a breakdown for some *F*

Conclusions

- In regions with no continental boundaries topography/topographic form stress plays a crucial role in setting up the large-scale oceanic currents.
- We demonstrated that quasi-geostrophic theory, even with a simple 1-layer model, can capture the existence of an eddy-saturation regime.
- We derived a bounds based on energy/enstrophy constraint for the form stress.
- We have seen that as the wind stress increases the momentum imparted by the ocean is balanced mostly by the form stress and only little by bottom drag… until a threshold wind value is reached (“crisis”) when form stress breaks down and get very large U in order to get balance.