Emergence of large-scale structure in planetary turbulence as an instability of the of the homogeneous turbulent state

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joint work with
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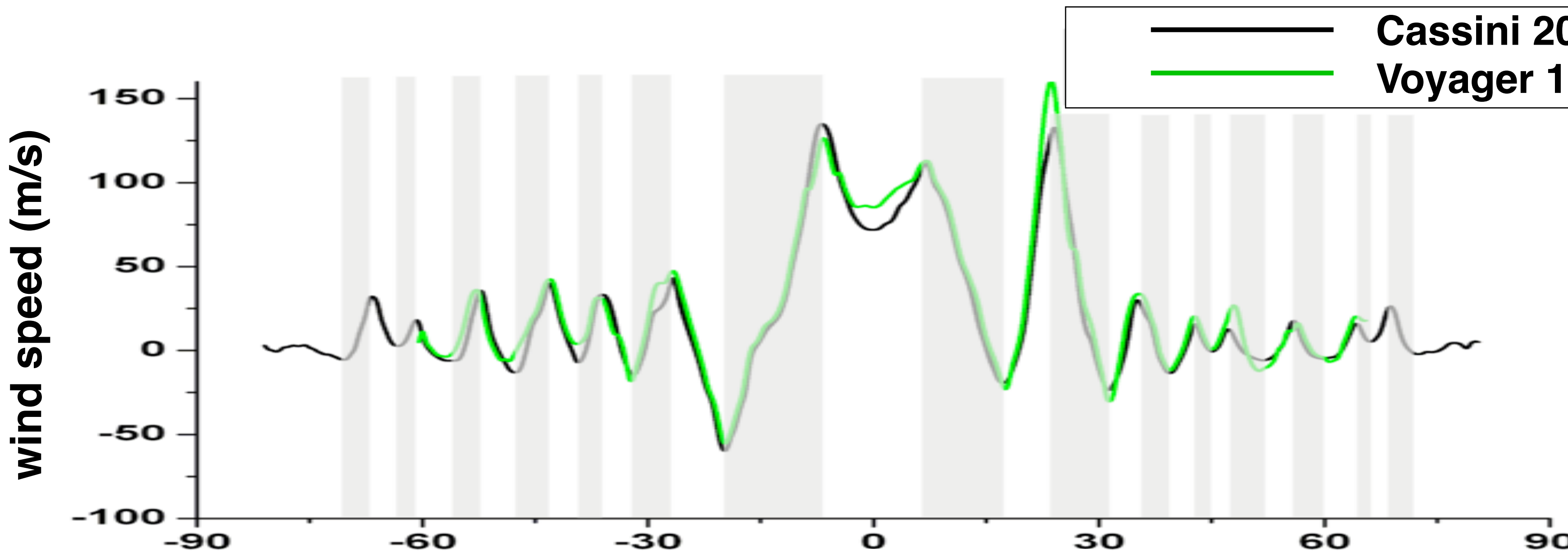
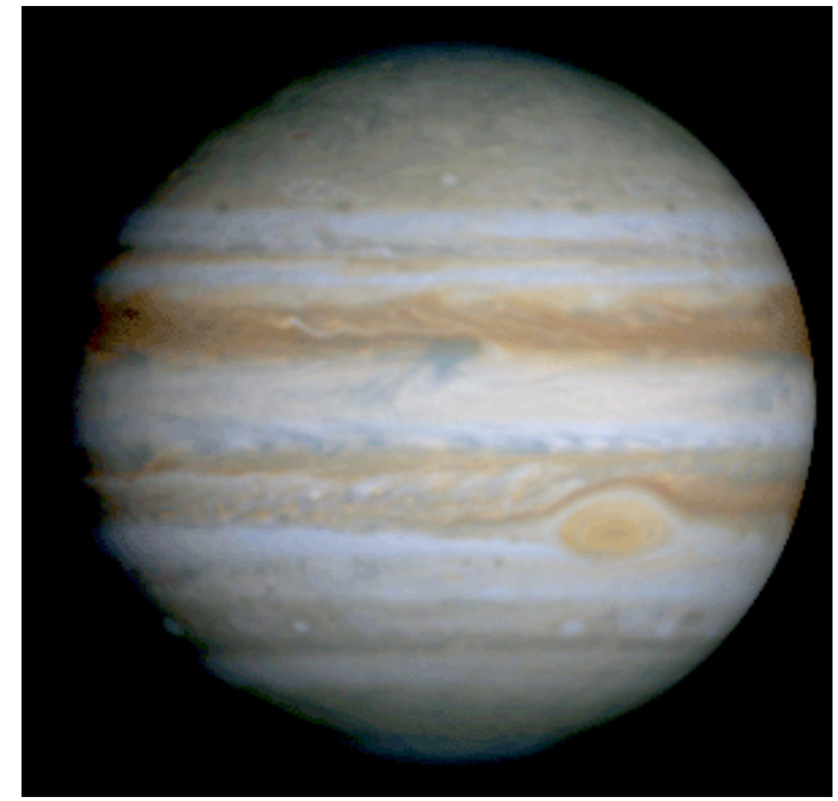
main points

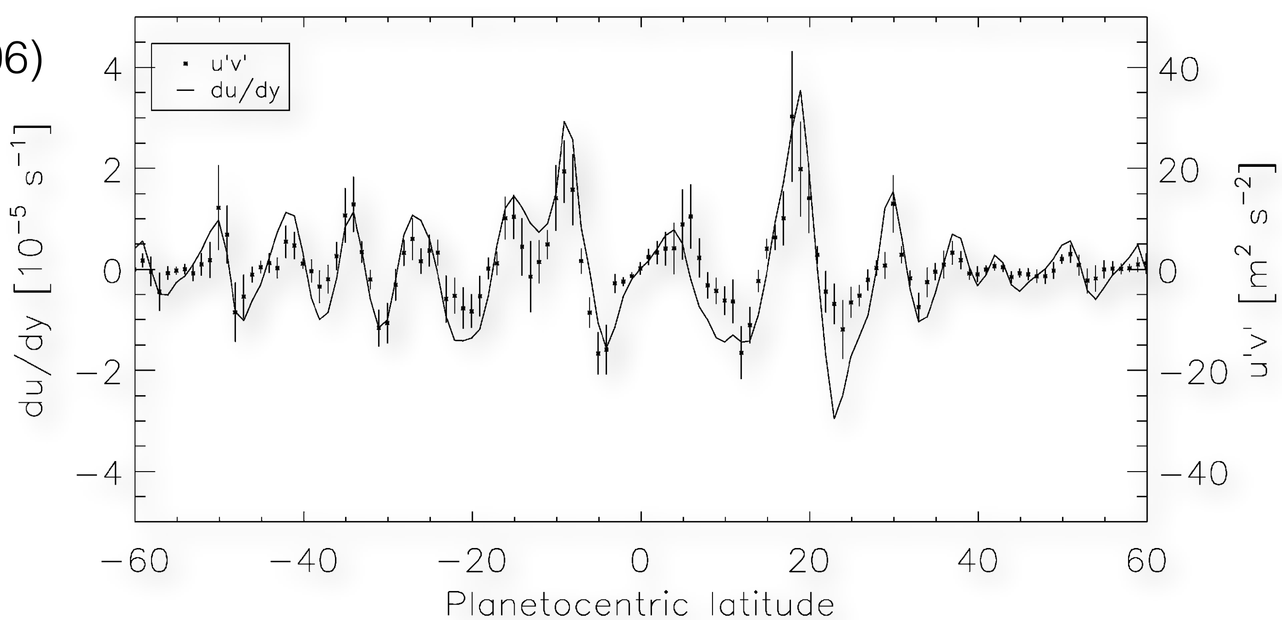
- turbulence acts anti-diffusively maintaining large-scale jets (Jupiter's winds, Earth's polar jet-stream)
  [this is known for finite amplitude jets]
- S3T describes the joint dynamics of the mean flow and the eddy statistics (closed at second order)
- turbulence acts anti-diffusively reinforcing ***even infinitesimal*** amplitude jets (leading to instability)
- modulational instability of Rossby waves is a special case of S3T instability
- within S3T we can study the statistical stability of inhomogeneous (i.e. with finite amplitude jets) turbulent statistical equilibria

Jets are eddy-driven





(Salyk et. al. 2006)



$$\overline{u'v'} = \kappa \frac{\partial U}{\partial y}$$

$$\kappa \approx 10^6 \ \mathrm{m^2\,s^{-1}}$$

$$\partial_t U = -\partial_y \overline{u'v'} = -\kappa \frac{\partial^2 U}{\partial y^2}$$

anti-diffusion!!

Barotropic vorticity equation on a beta-plane

$$\partial_t \zeta + \mathbf{u}\cdot\nabla\zeta + \beta v = -r\zeta + \sqrt{\varepsilon}\xi$$

dissipation

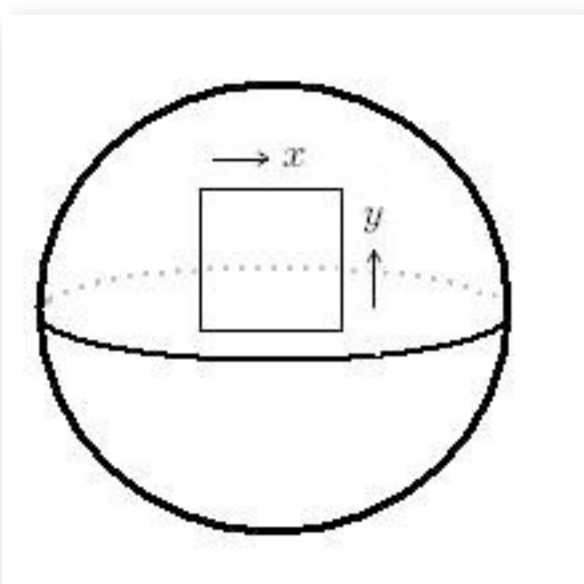
stochastic forcing

$$\nabla\cdot\mathbf{u} = 0$$

$$\mathbf{u} = \hat{\mathbf{z}}\times\nabla\psi$$

$$\nabla\times\mathbf{u} = \underbrace{\Delta\psi}_{\zeta}\ \hat{\mathbf{z}}$$

$$\langle \xi(\mathbf{x}_a,t)\xi(\mathbf{x}_b,t')\rangle = Q(\mathbf{x}_a-\mathbf{x}_b)\,\delta(t-t')$$



we non-dimensionalize using

time scale : $1/r$

length scale : $L_f = 1/k_f$

The S3T dynamical system

$$Z(\mathbf{x},t) = \langle \zeta(\mathbf{x},t) \rangle \quad , \quad \zeta'(\mathbf{x},t) = \zeta(\mathbf{x},t) - Z(\mathbf{x},t)$$
$$C(\mathbf{x}_a,\mathbf{x}_b,t) = \langle \zeta'(\mathbf{x}_a,t)\zeta'(\mathbf{x}_b,t) \rangle$$

restrict nonlinearity by not allowing
eddy-eddy → eddy interactions (**QL**)

$$\partial_t Z + \mathbf{U}\cdot\nabla Z + \beta V = \mathcal{R}(C) - Z$$
$$\partial_t C_{ab} = \left[\mathcal{A}_a(\mathbf{U}) + \mathcal{A}_b(\mathbf{U})\right] C_{ab} + \varepsilon\, Q_{ab}$$

$$\mathcal{A}(\mathbf{U}) = -\mathbf{U}\cdot\nabla - \left[(\beta\hat{\mathbf{x}} - \Delta\mathbf{U})\cdot\nabla\right]\Delta^{-1} - 1$$
$$\mathcal{R}(C) = -\nabla\cdot\left[\frac{\hat{\mathbf{z}}}{2}\times(\nabla_a\Delta_a^{-1} + \nabla_b\Delta_b^{-1})C_{ab}\right]_{a=b} = -\nabla\cdot\langle \mathbf{u}'\zeta'\rangle$$

stability of homogeneous S3T equilibrium

$$\mathbf{U}^e = 0 \quad , \quad C^e(\mathbf{x}_a - \mathbf{x}_b) = \frac{\varepsilon\, Q}{2}$$

(for any ε, β)

perturbations $\delta Z,\ \delta C$ about the homogeneous equilibrium satisfy the linearized S3T equations:

$$\partial_t \delta Z = \mathcal{A}^e\, \delta Z + \mathcal{R}(\delta C)$$

$$\partial_t \delta C_{ab} = (\mathcal{A}^e_a + \mathcal{A}^e_b)\delta C_{ab} + (\delta\mathcal{A}_a + \delta\mathcal{A}_b) C^e_{ab}$$

$$\mathcal{A}^e \equiv \mathcal{A}(\mathbf{U}^e)$$

eigenfunctions:

$$\delta Z = \underbrace{e^{\mathrm{i}\mathbf{n}\cdot\mathbf{x}}}_{\delta\tilde{Z}}\, e^{(\sigma - \mathrm{i}\omega_{\mathbf{n}})t}$$

(plane wave)

ω: Rossby wave frequency

$$\omega_{\mathbf{n}} = \frac{-\beta n_x}{n^2}$$

$$\delta C_{ab} = \underbrace{C^{(\mathrm{h})}_{\mathbf{n}}(\mathbf{x}_a - \mathbf{x}_b) e^{\mathrm{i}\mathbf{n}\cdot(\mathbf{x}_a + \mathbf{x}_b)/2}}_{\delta\tilde{C}_{ab}}\ e^{(\sigma - \mathrm{i}\omega_{\mathbf{n}})t}$$

because $\delta\tilde{Z}$ is an eigenfunction of $\mathcal{A}^e$ we have that $\longrightarrow$ $\mathcal{R}(\delta\tilde{C}) = \varepsilon f(\sigma)\, \delta\tilde{Z}$

eigenvalue relation for the stability of homogeneous S3T equilibrium

for given ε, β and $\hat{Q}(\mathbf{k})$, eigenvalue that corresponds to eigenfunction with wavevector **n** satisfies:

$$\sigma + 1 = \varepsilon \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \frac{|\mathbf{n}\times\mathbf{k}|^2 (k_s^2 - k^2)(k^2 - n^2)}{k_s^2 k^4 n^2 \left[\sigma + 2 + \mathrm{i}\left(\omega_{\mathbf{k}+\mathbf{n}} - \omega_{\mathbf{k}} - \omega_{\mathbf{n}}\right)\right]} \frac{\hat{Q}(\mathbf{k})}{2}$$

$$\hat{Q}(\mathbf{k}) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} Q(\mathbf{x}_a - \mathbf{x}_b) e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_a - \mathbf{x}_b)}$$

$$\mathbf{k}_s = \mathbf{k} + \mathbf{n} \ \ , \ \ k_s = |\mathbf{k}_s|$$

ω: Rossby wave frequency

$$\omega_{\mathbf{n}} = \frac{-\beta n_x}{n^2}$$

eigenvalue relation for the stability of homogeneous S3T equilibrium

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$$\hat{Q}(\mathbf{k}) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} Q(\mathbf{x}_a - \mathbf{x}_b) e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_a - \mathbf{x}_b)}$$

$$\mathbf{k}_s = \mathbf{k} + \mathbf{n} \quad , \quad k_s = |\mathbf{k}_s|$$

ω: Rossby wave frequency

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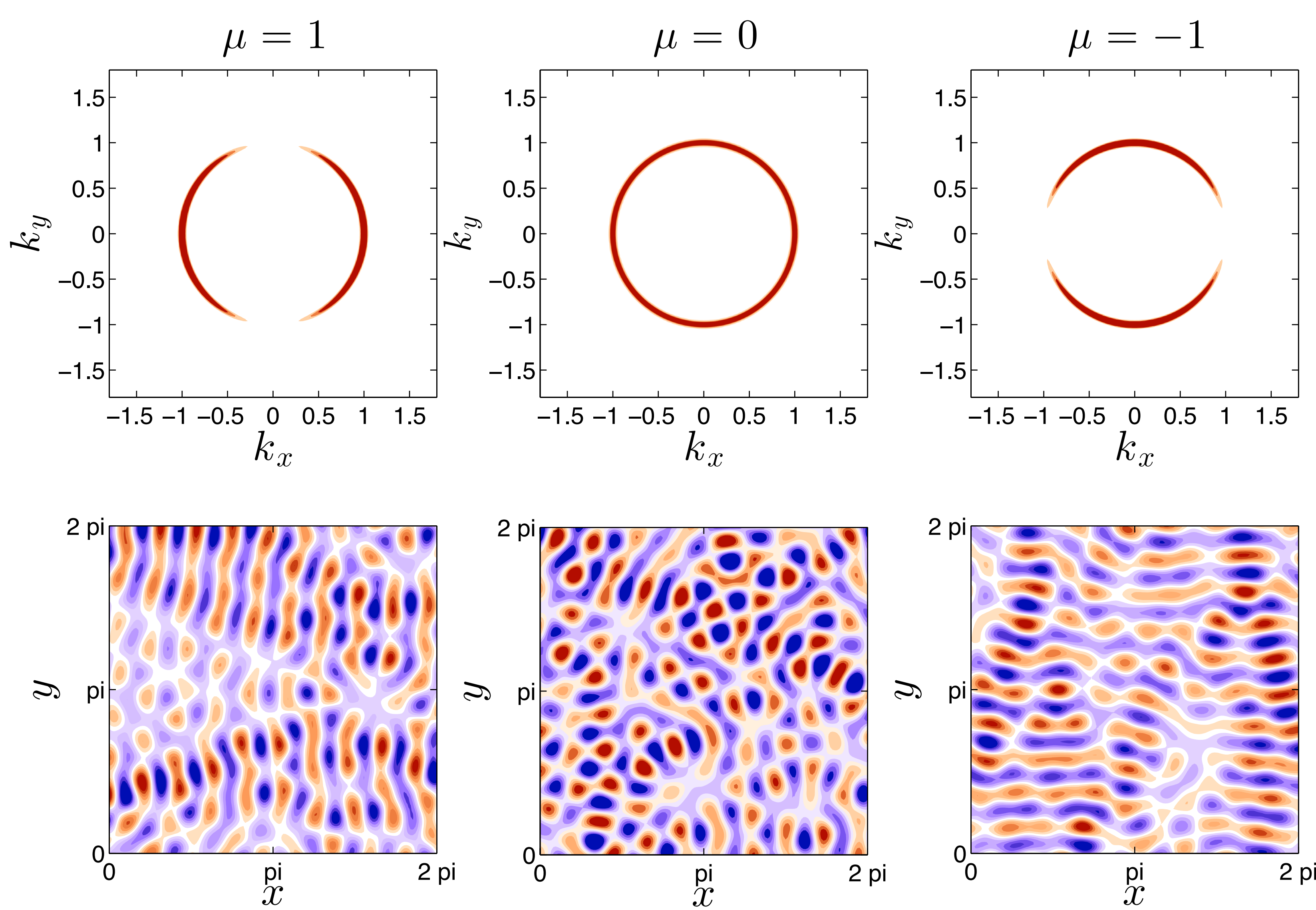
take forcing structure

$$\hat{Q}(\mathbf{k}) \sim \delta(k - k_f) \left[1 + \mu \frac{k_x^2 - k_y^2}{k^2}\right]$$

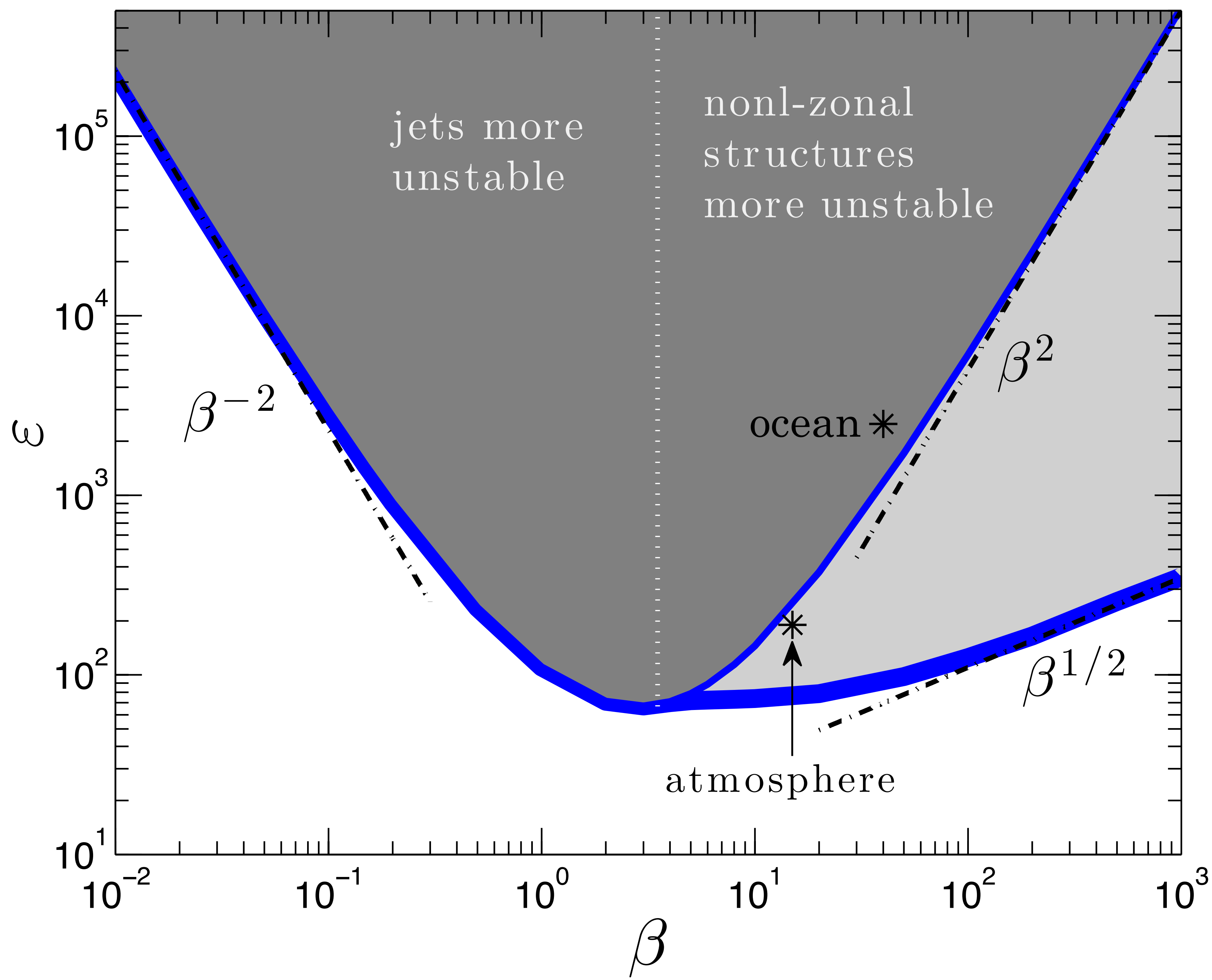
anisotropic (baroclinic) | isotropic | anisotropic

$\hat{Q}(\mathbf{k})$

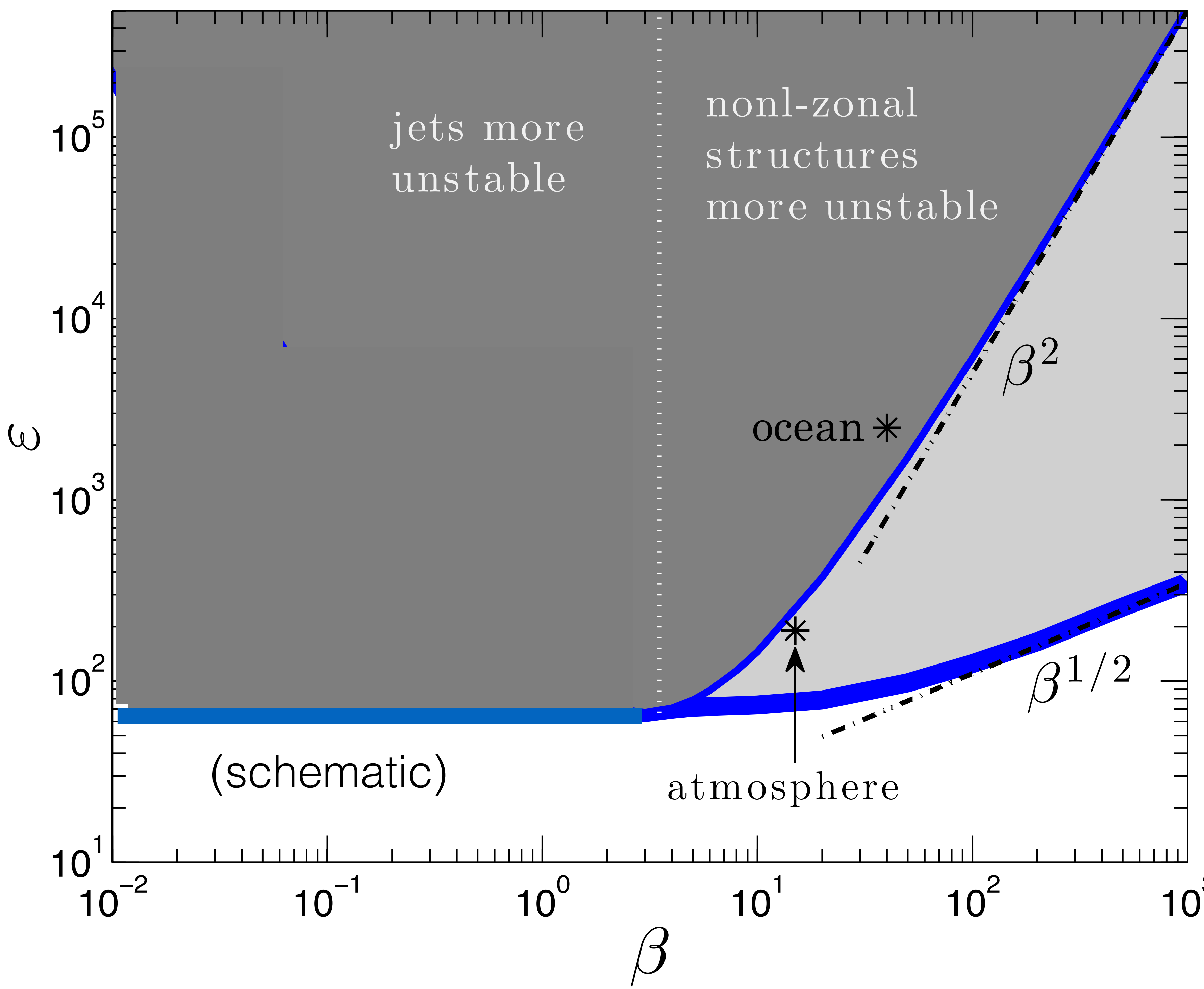
$\xi(\mathbf{x}, t)$



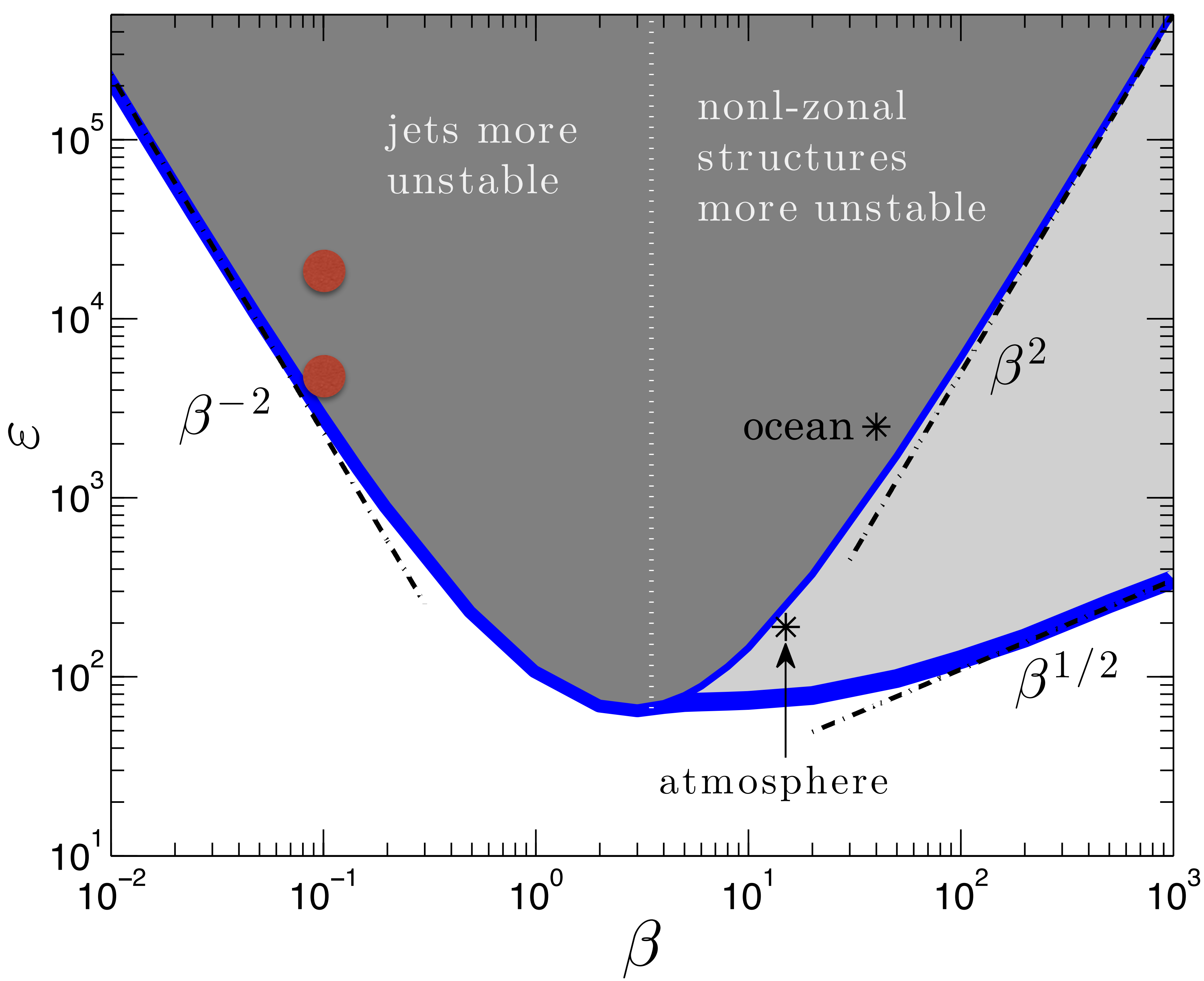
Critical ε for S3T instability with isotropic forcing



Critical ε for S3T instability with anisotropic forcing



small β regime



small β regime

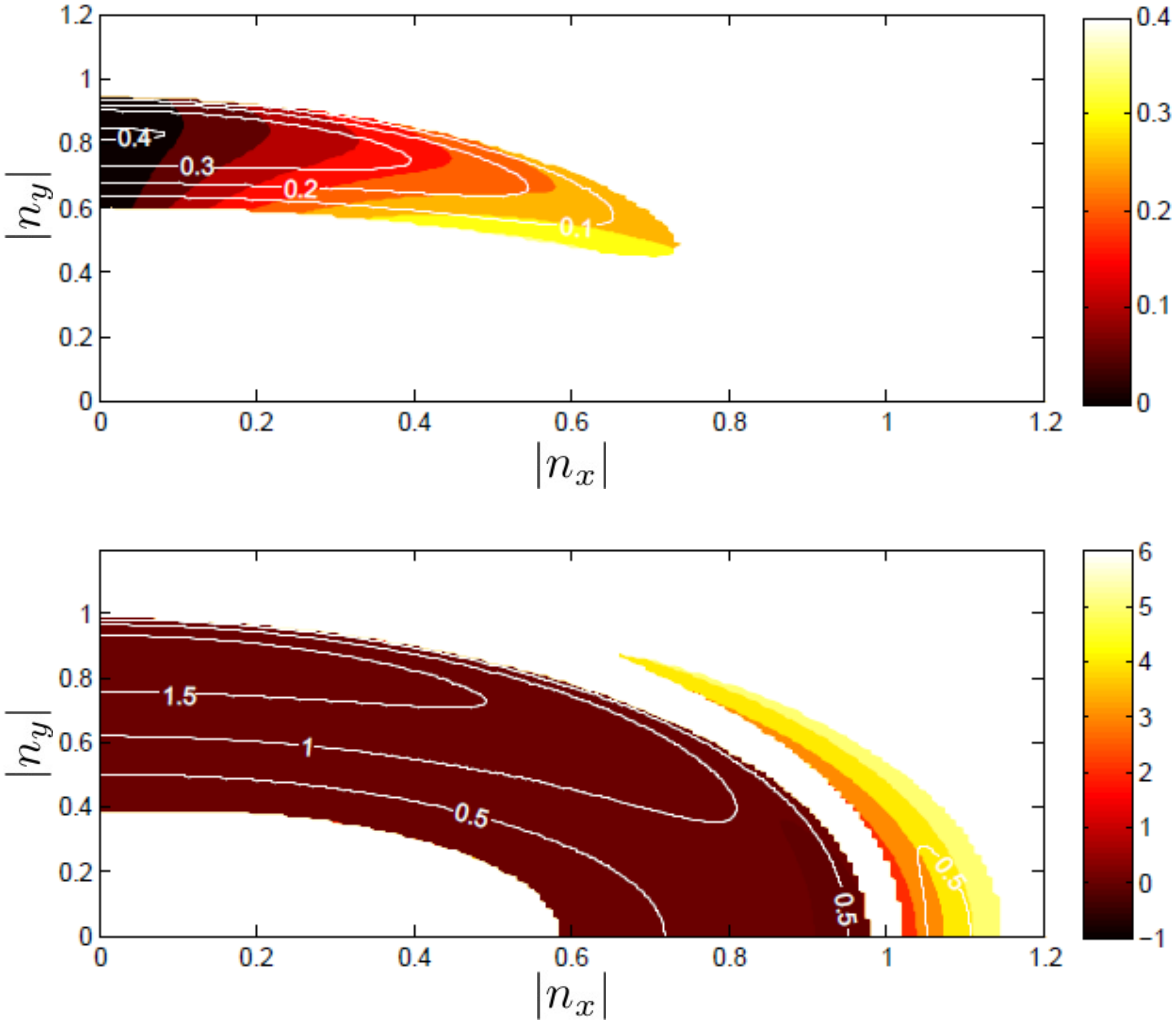
$\beta = 0.1$

$\varepsilon = 2\varepsilon_c$

(remember wavenumbers are scaled with kf)

$\beta = 0.1$

$\varepsilon = 10\varepsilon_c$

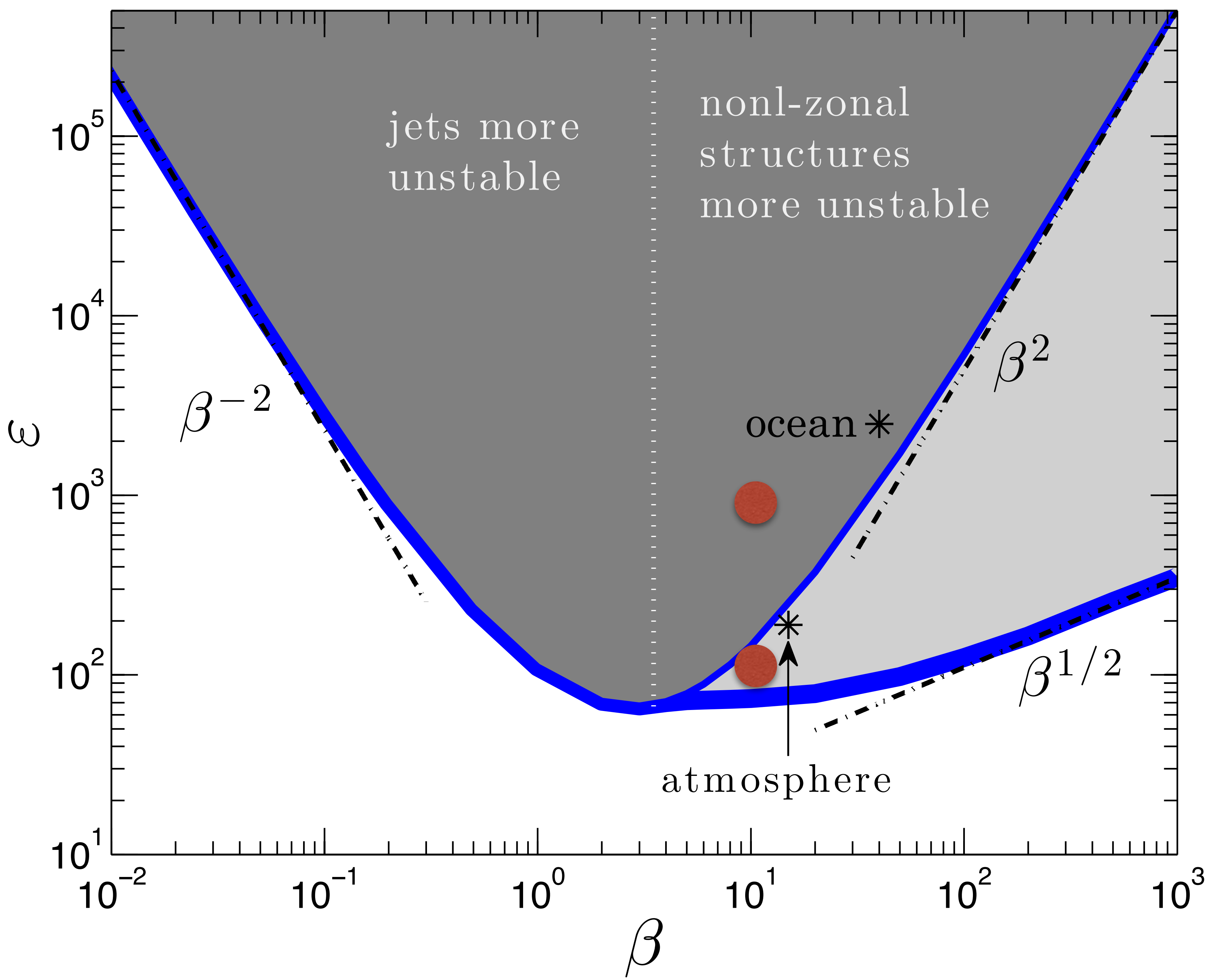


contours: growth-rate

color shading: phase speed

jets are the most unstable

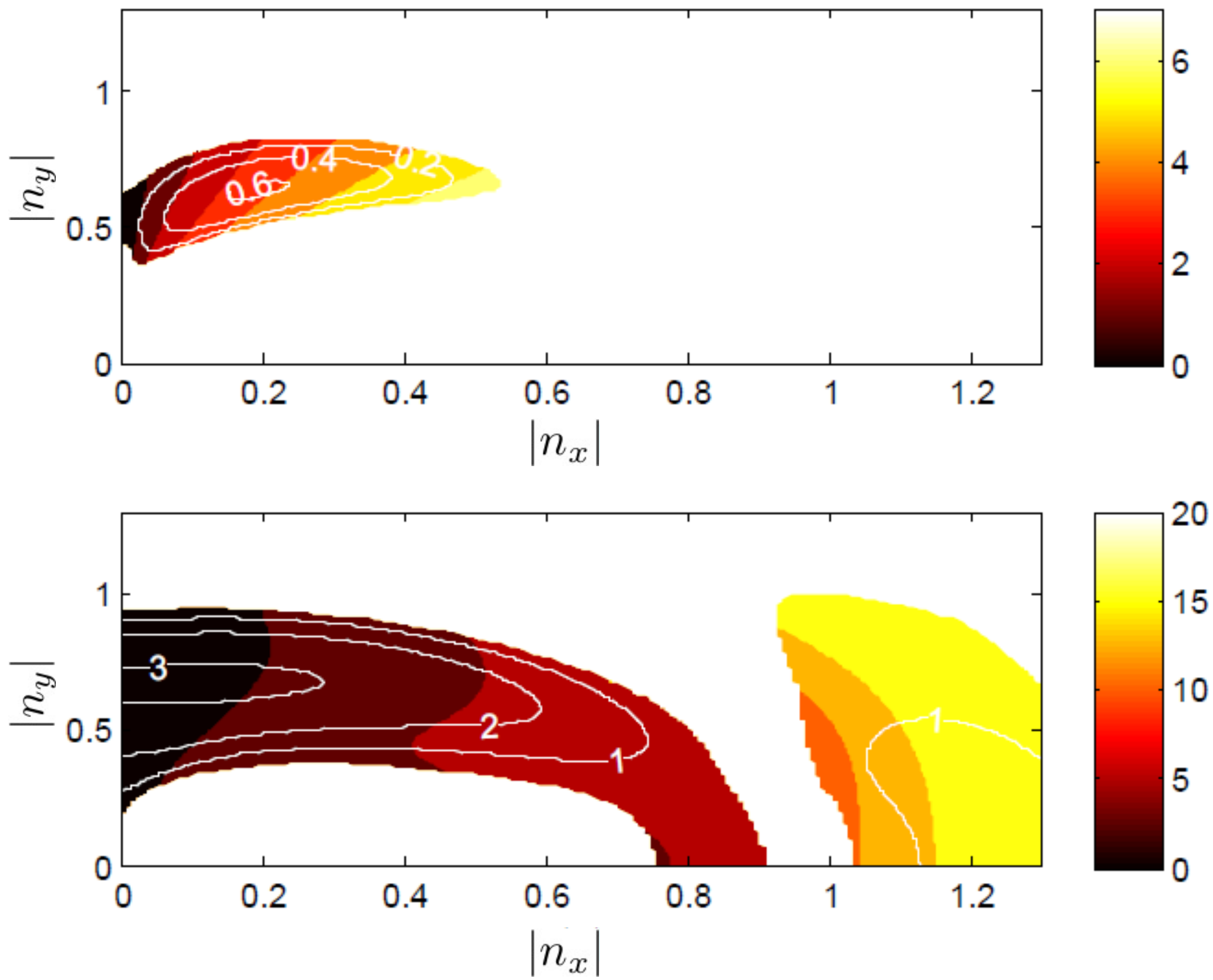
large β regime



large β regime

$\beta = 10$
$\varepsilon = 2\varepsilon_c$

$\beta = 10$
$\varepsilon = 10\varepsilon_c$



contours: growth-rate

color shading: phase speed

non-zonal structures are the most unstable (westward propagating)

We want to study the eddy—mean flow dynamics of the S3T instability near the stability boundary

$$\sigma + 1 = \varepsilon f(\sigma) \quad \Rightarrow \quad \mathrm{Re}(\sigma) + 1 = \varepsilon \mathrm{Re}[f(\sigma)]$$

There is instability if Re(σ)>0 which can occur for an appropriate ε only if Re[f(σ)]>0.

At the stability boundary $\varepsilon = \varepsilon_c$

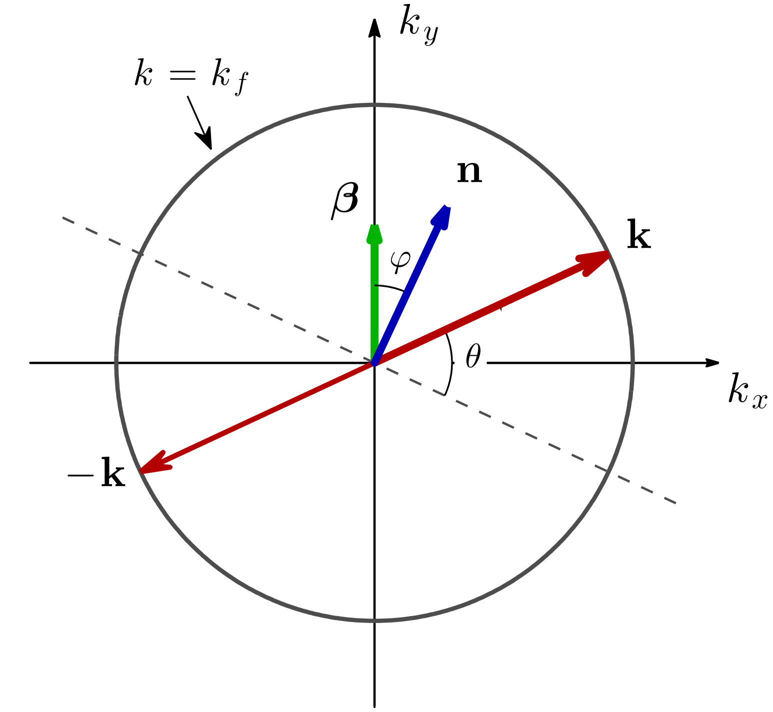
we set $\sigma = 0$ and therefore instability is controlled by:

$$\boxed{f_r \equiv \mathrm{Re}[f(0)]}$$

for ring forcing in wavenumber space

f_r is expressed as a sum over the spectral components of the forcing

$$f_r = \int_0^\pi \underbrace{\left[F(\mathbf{n},\theta) + F(\mathbf{n},\pi+\theta)\right]}_{\mathcal{F}(\mathbf{n},\theta)} \mathrm{d}\theta$$

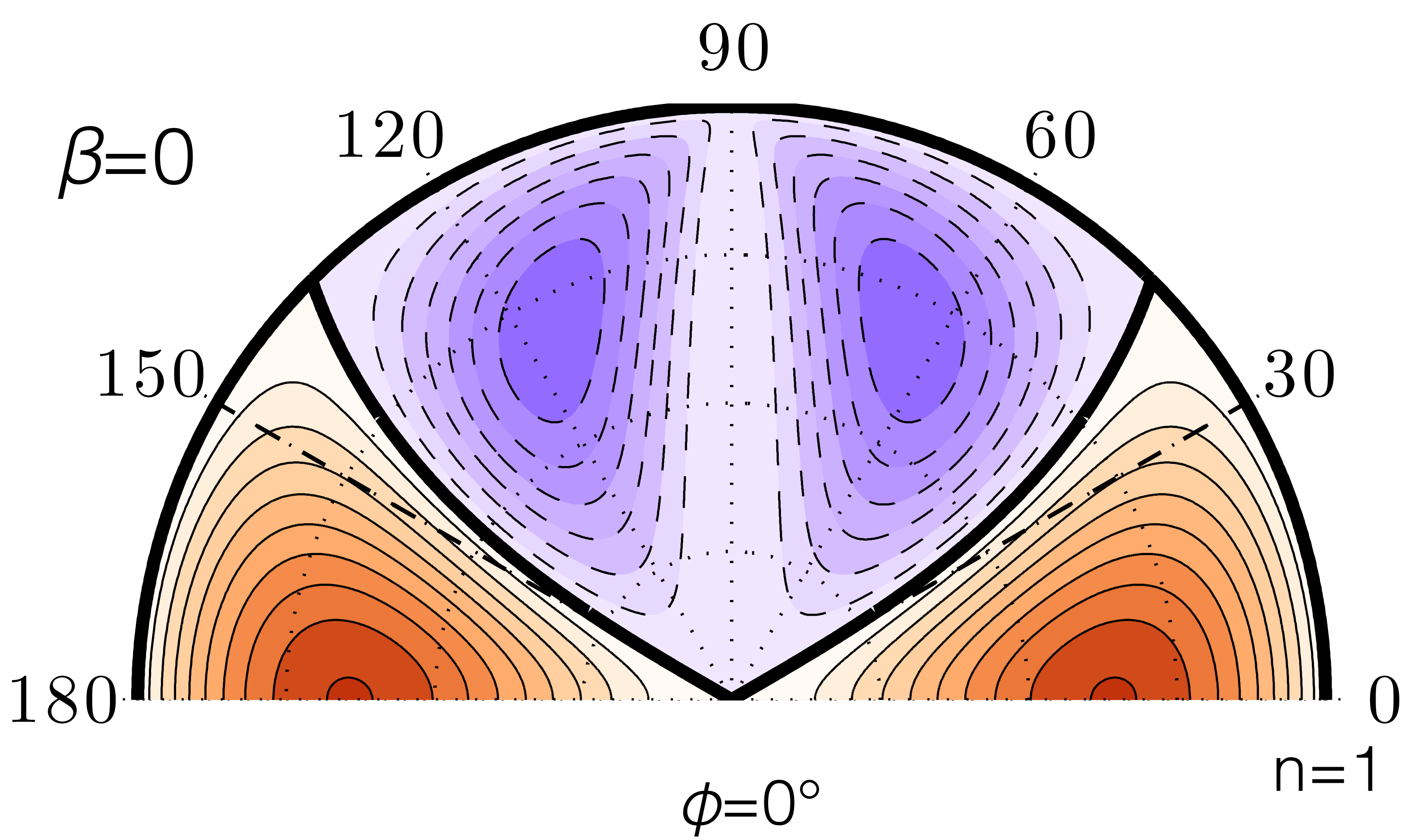


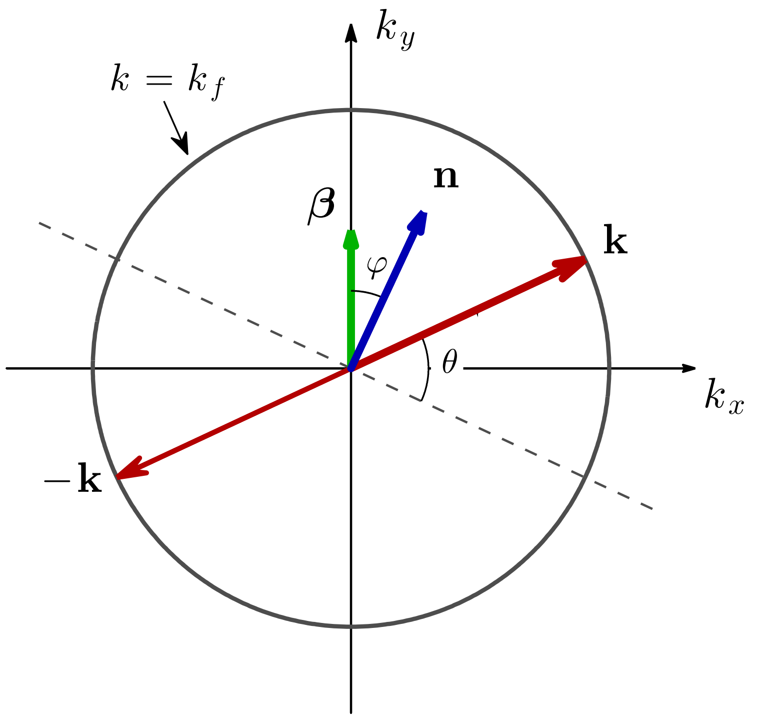
$$\mathbf{k} = (\cos\theta, \sin\theta)$$

$$\mathbf{n} = (n\sin\varphi, n\cos\varphi)$$

small β

Contribution to f_r (i.e. contours of $\mathcal{F}(\mathbf{n},\theta)$) in (n,θ) polar plot





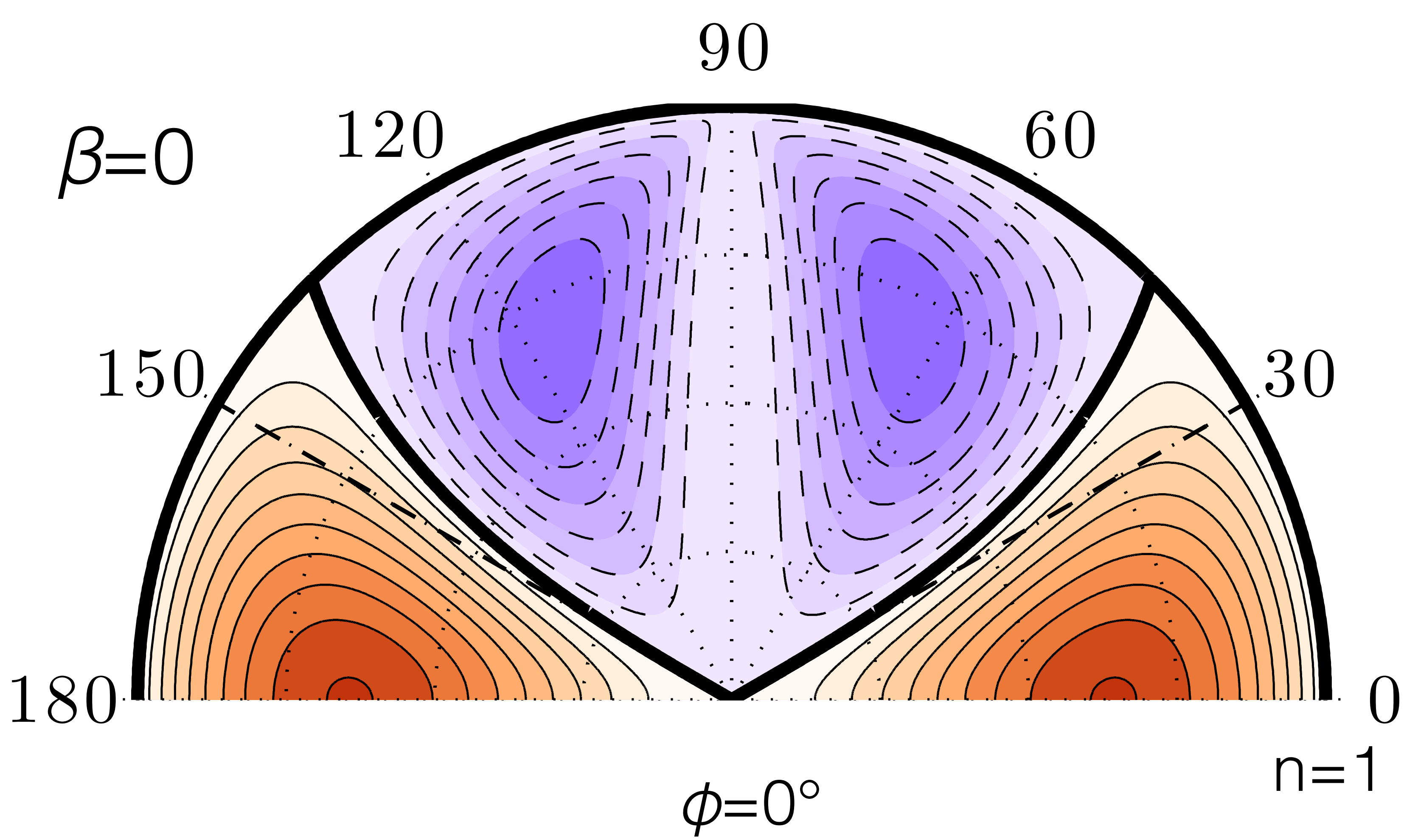
$$\mathbf{k} = (\cos\theta, \sin\theta)$$
$$\mathbf{n} = (n\sin\varphi, n\cos\varphi)$$

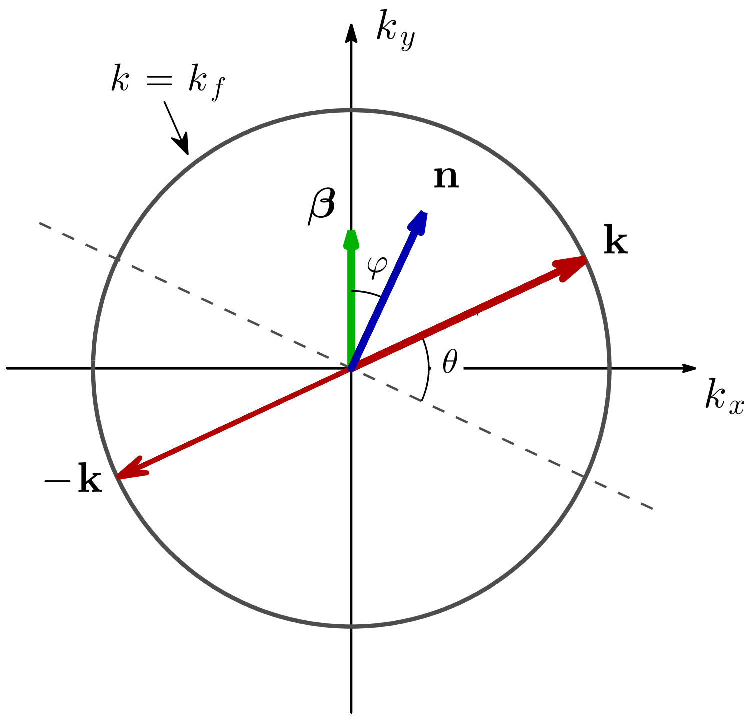
$$f_r = \mathrm{Re}\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\, \frac{|\mathbf{n}\times\mathbf{k}|^2(k_s^2 - k^2)(k^2 - n^2)}{k_s^2 k^4 n^2\,[2 + \mathrm{i}\,(\omega_{\mathbf{k}+\mathbf{n}} - \omega_{\mathbf{k}} - \omega_{\mathbf{n}})]}\, \frac{\hat{Q}(\mathbf{k})}{2}$$

$$\omega_{\mathbf{n}} = \frac{-\beta n_x}{n^2}$$

small β

Contribution to f_r (i.e. contours of $\mathcal{F}(\mathbf{n},\theta)$) in (n,θ) polar plot





$$\mathbf{k} = (\cos\theta, \sin\theta)$$

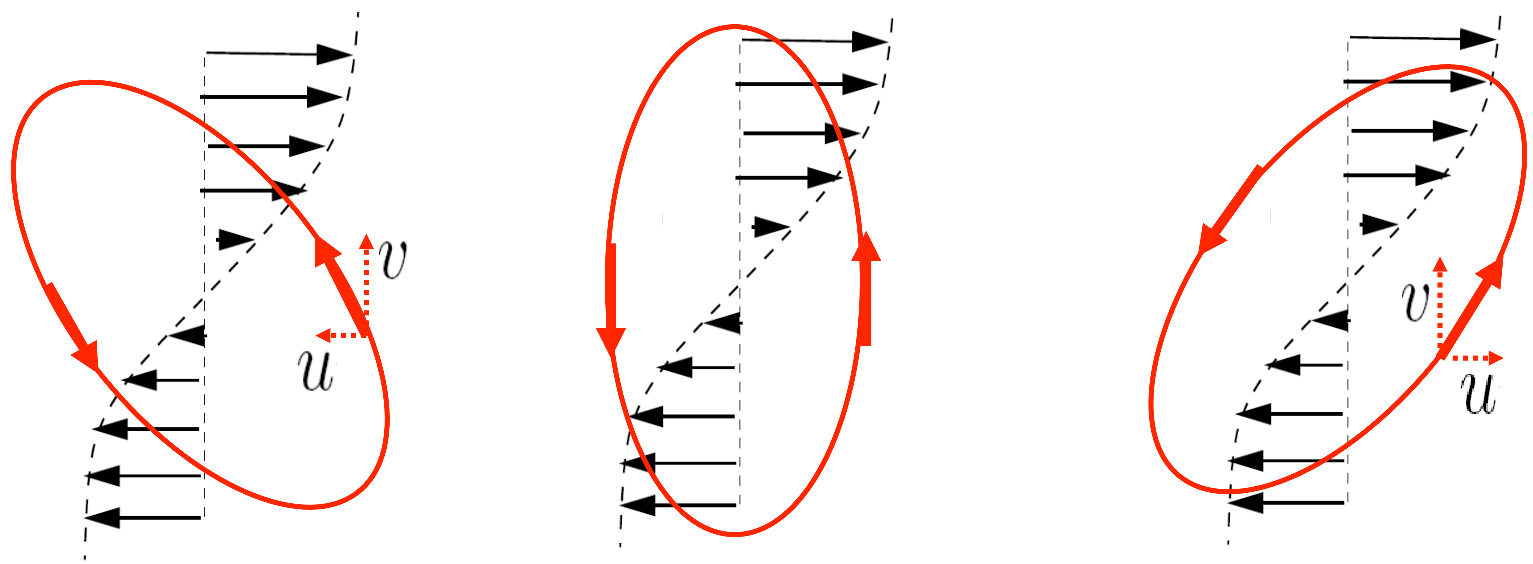
$$\mathbf{n} = (n\sin\varphi, n\cos\varphi)$$

contribution to f_r from wide range of **k**'s
all **k**'s with |θ|<30 contribute

$$\omega_{\mathbf{n}} = \frac{-\beta n_x}{n^2}$$

Orr mechanism

finite shear flow



$\overline{u'v'} > 0$

eddies grow

mean flow decreases

$\overline{u'v'} < 0$

eddies decay

mean flow increases

turning time proportional to (mean flow shear)$^{-1}$

Orr mechanism

infinitesimal shear?

shear time»dissipation time ⟶ eddies don’t manage to shear over all the way

what matters then is what the eddies do instantaneously

θ<30° eddies instantaneously give momentum flux to the mean flow

θ>30° eddies instantaneously give momentum flux to the mean flow

turbulence acts anti-diffusively even with infinitesimal mean flow

anisotropic forcing (μ ≠ 0)

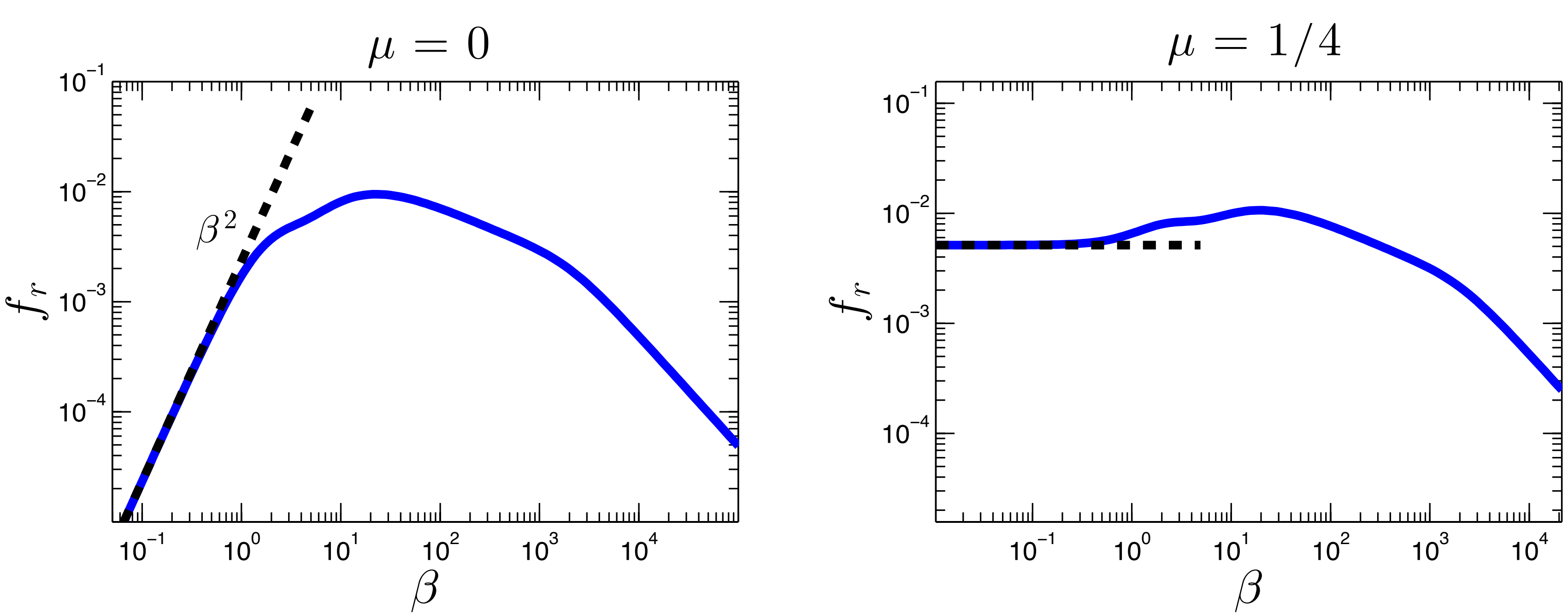
$$-\partial_y \, \delta\langle u'v'\rangle = -\frac{\mu}{8}\cos(2\varphi)\,\partial^2_{yy}\delta U$$

2nd order anti-diffusion

isotropic forcing (μ = 0)

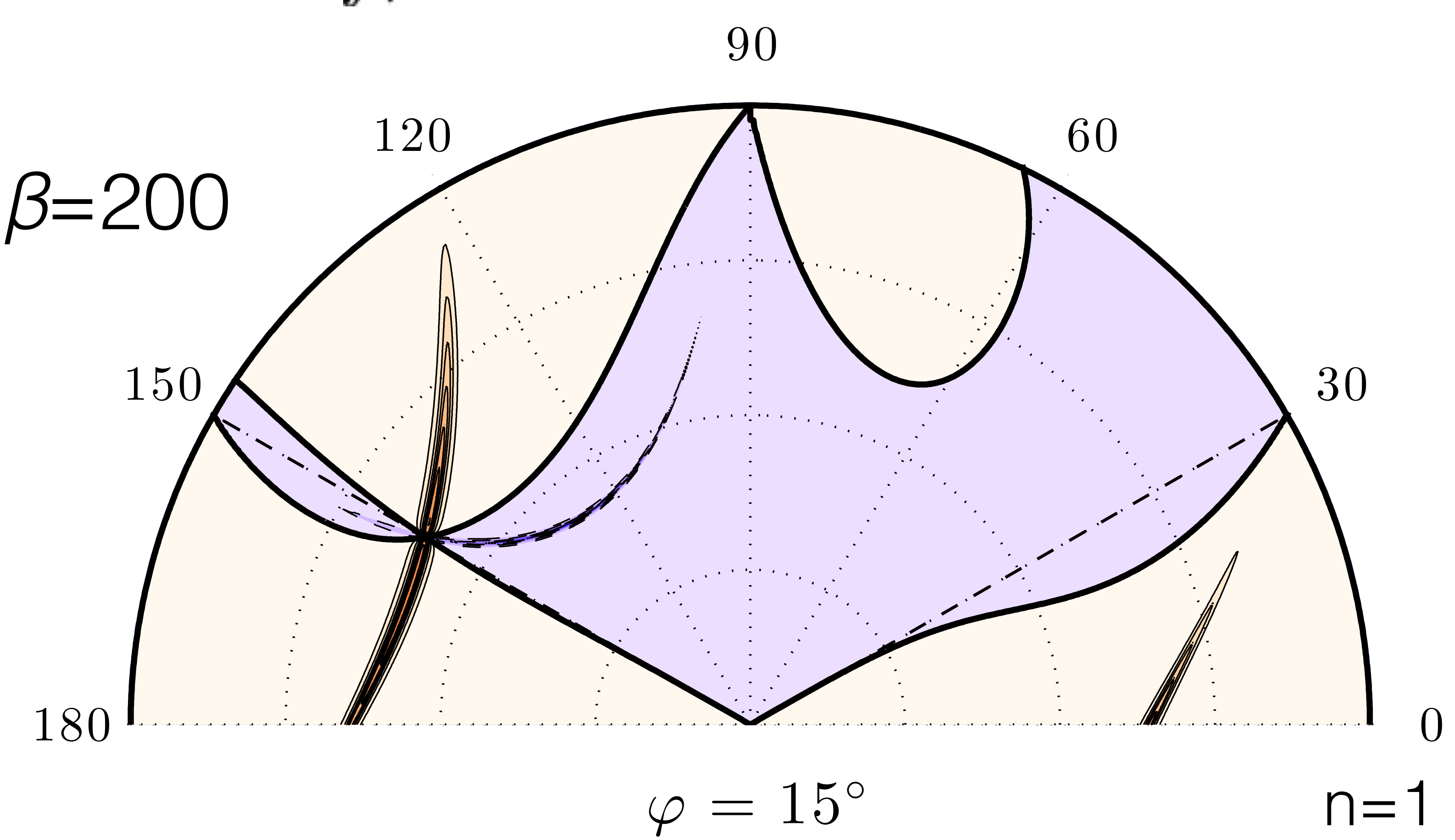
$$-\partial_y \, \delta\langle u'v'\rangle = \frac{\beta^2}{64}[2+\cos(2\varphi)]\,\partial^4_{yyyy}\delta U$$

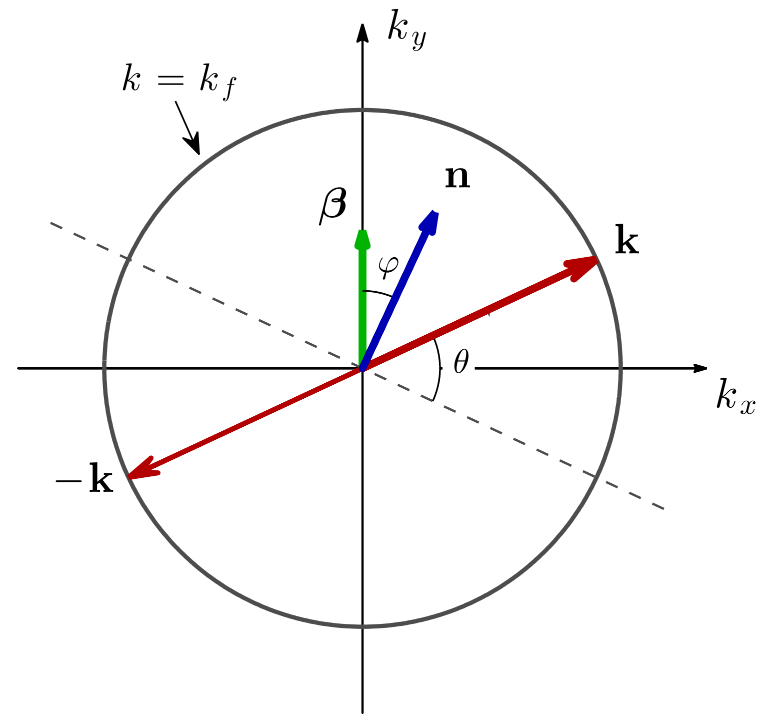
4th order hyper-anti-diffusion



large β

Contribution to f_r (i.e. contours of $\mathcal{F}(\mathbf{n},\theta)$) in (n,θ) polar plot





$$\mathbf{k} = (\cos\theta, \sin\theta)$$
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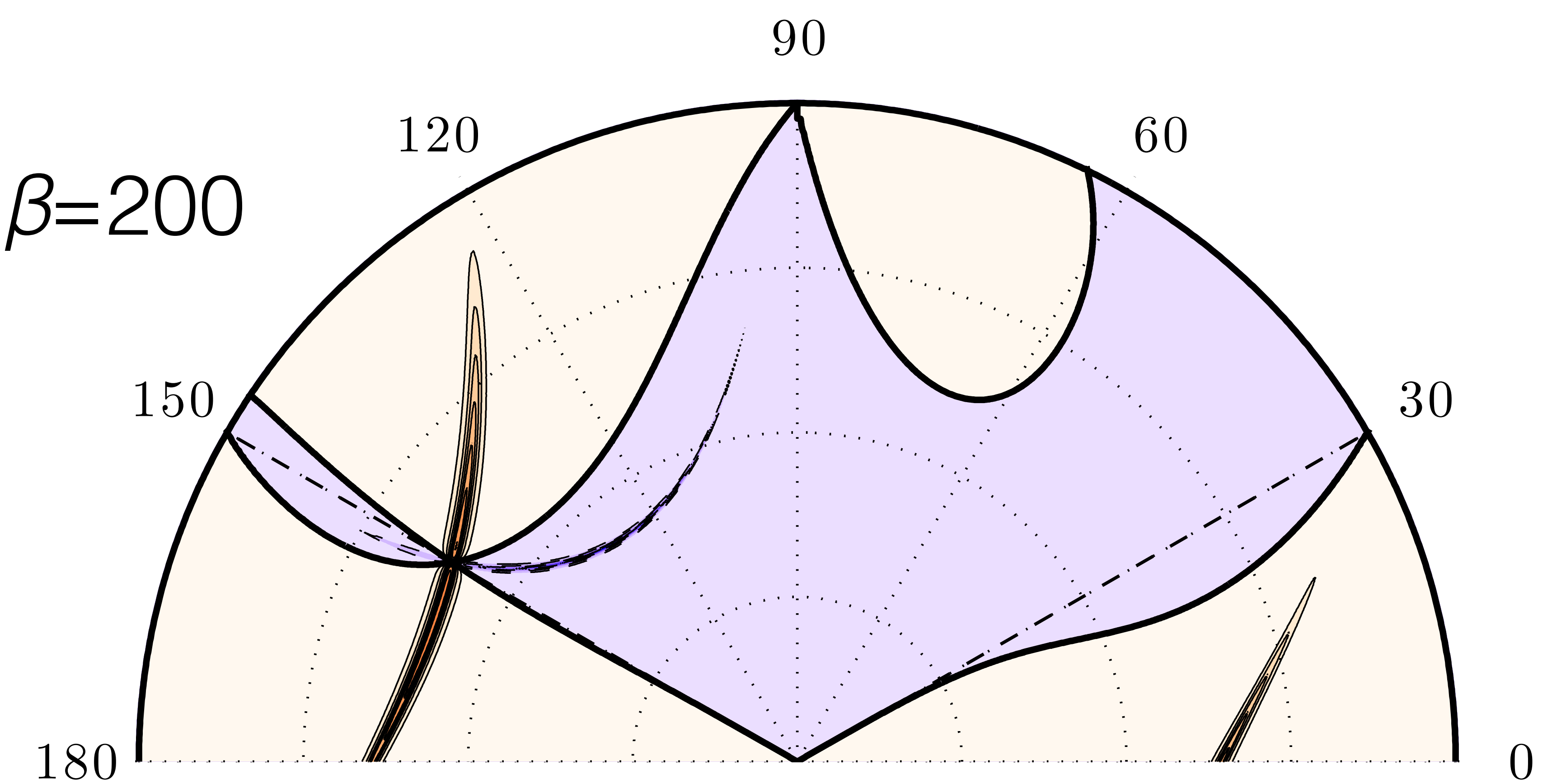
$$f_r = \mathrm{Re}\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \frac{|\mathbf{n}\times\mathbf{k}|^2(k_s^2-k^2)(k^2-n^2)}{k_s^2 k^4 n^2\left[2+\mathrm{i}\left(\omega_{\mathbf{k}+\mathbf{n}}-\omega_{\mathbf{k}}-\omega_{\mathbf{n}}\right)\right]} \frac{\hat{Q}(\mathbf{k})}{2}$$

$$\omega_{\mathbf{n}} = \frac{-\beta n_x}{n^2}$$

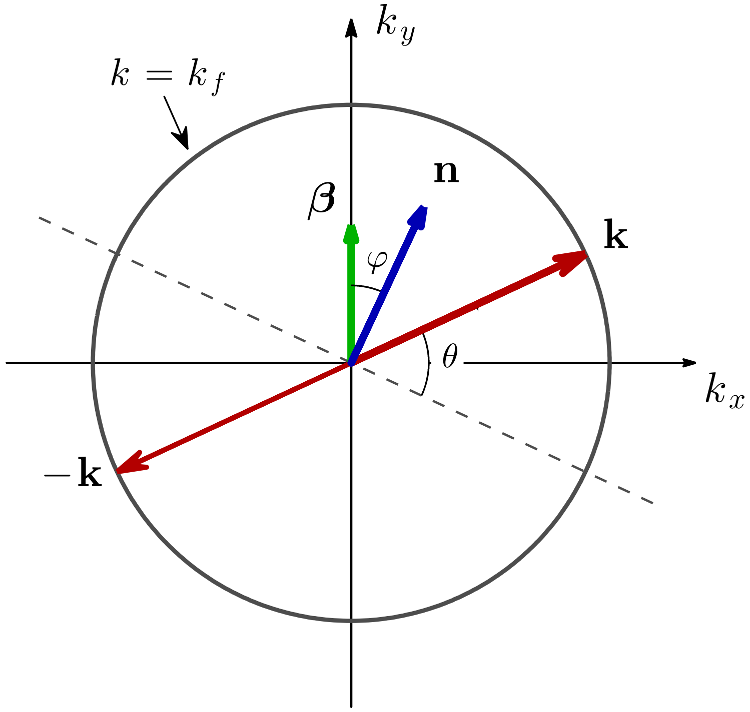
large β

Contribution to f_r (i.e. contours of $\mathcal{F}(\mathbf{n},\theta)$) in (n,θ) polar plot

β=200



$\varphi = 15^\circ$ n=1



$$\mathbf{k} = (\cos\theta, \sin\theta)$$

$$\mathbf{n} = (n\sin\varphi, n\cos\varphi)$$

contribution to f_r only from small range of **k**'s

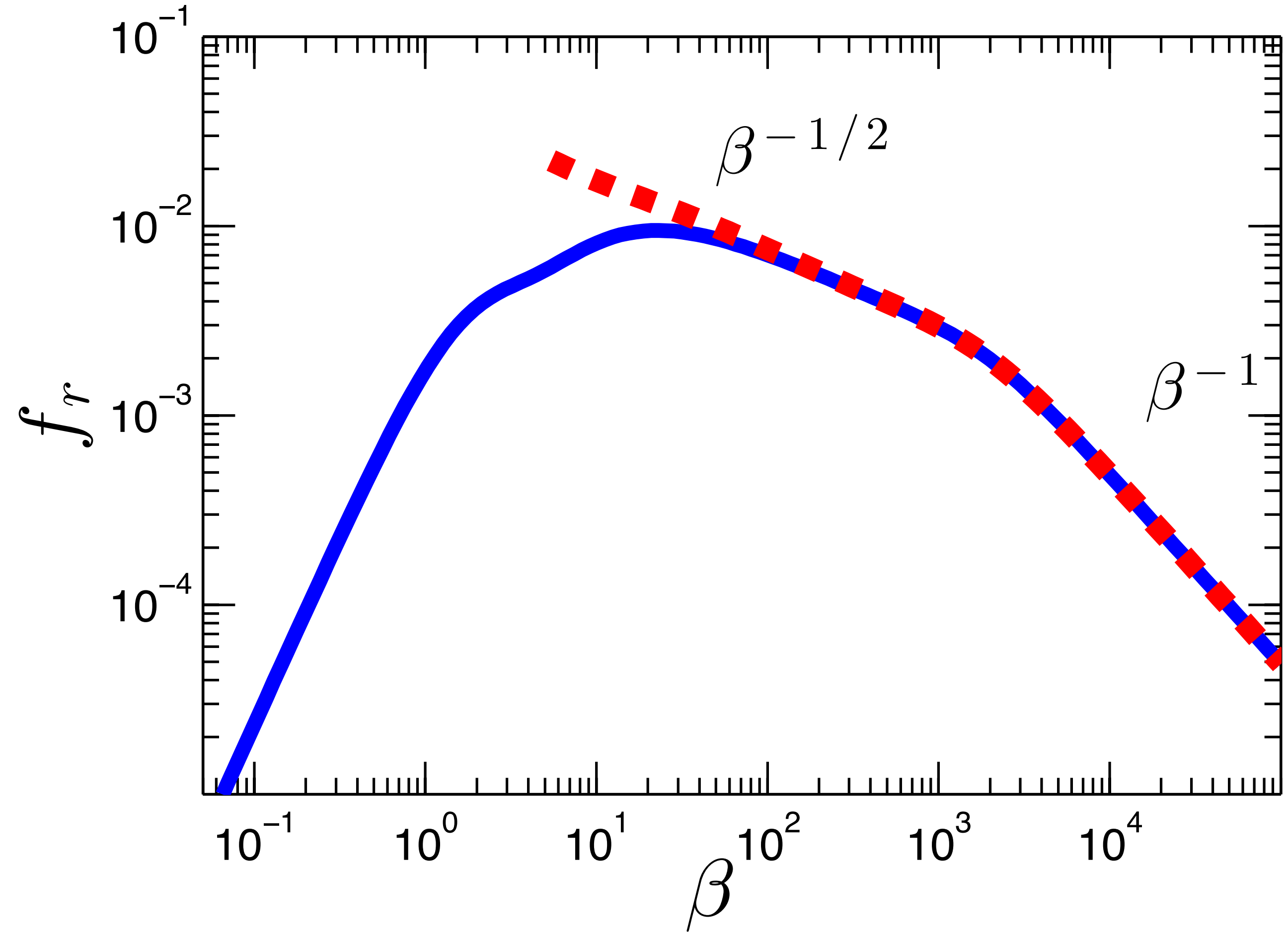
$$\omega_{\mathbf{n}} = \frac{-\beta n_x}{n^2}$$

Contribution to f_r for β≫1

for β≫1 the contribution to f_r reduces to the contribution only near the resonances

asymptotic expansion
for the resonant contribution

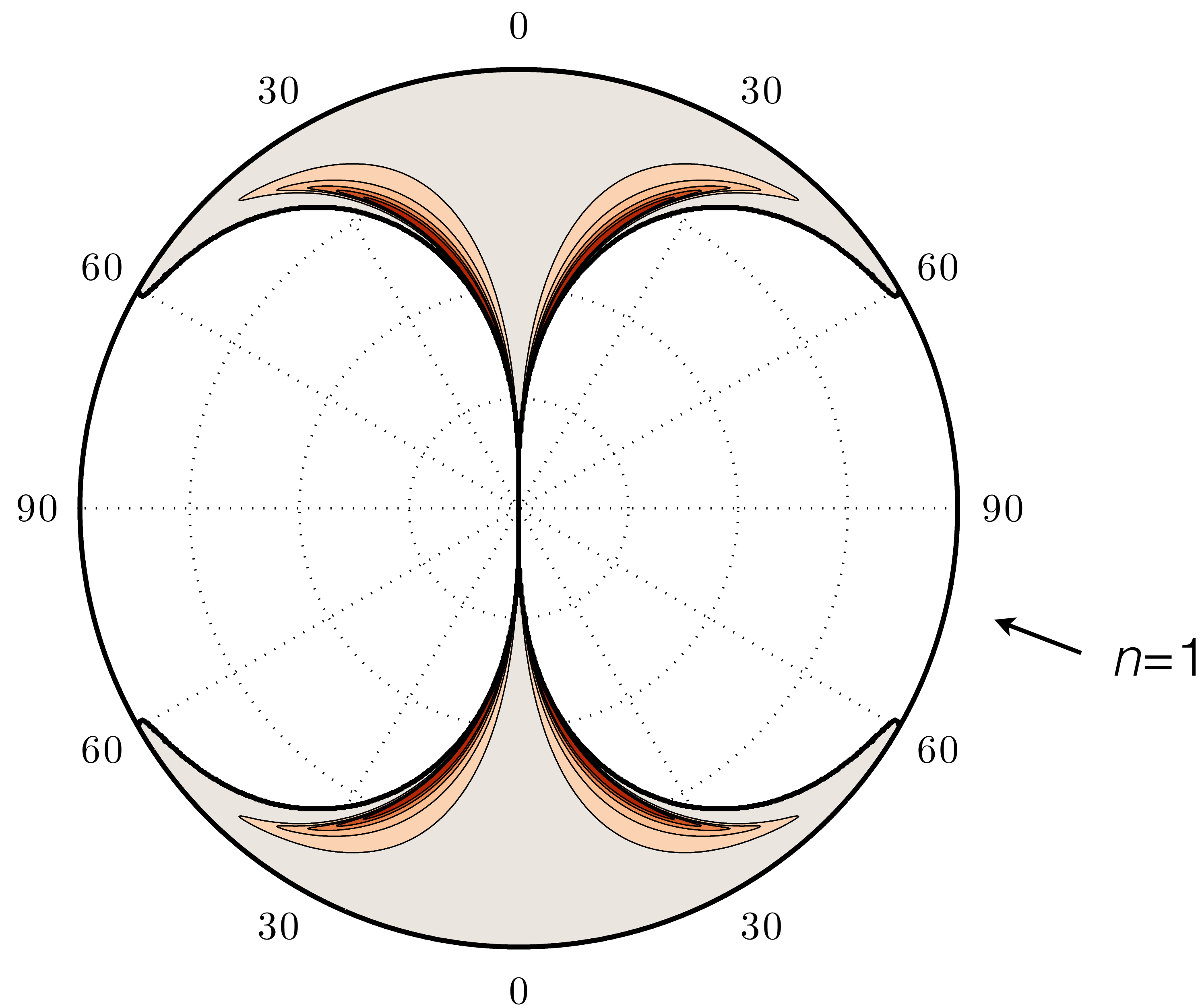
$$f_r^{(\mathrm{R})}(\mathbf{n}) = \frac{1}{\sqrt{\beta}} \sum_{j=1}^{N_r} \frac{\pi \mathcal{N}_j \, \eta_j}{\mathcal{D}_{0,j}^{1/2} |\lambda_j|^{1/2}}$$



(cf. Bakas, Constantinou & Ioannou 2014)

f_r expresses the tendency for instability

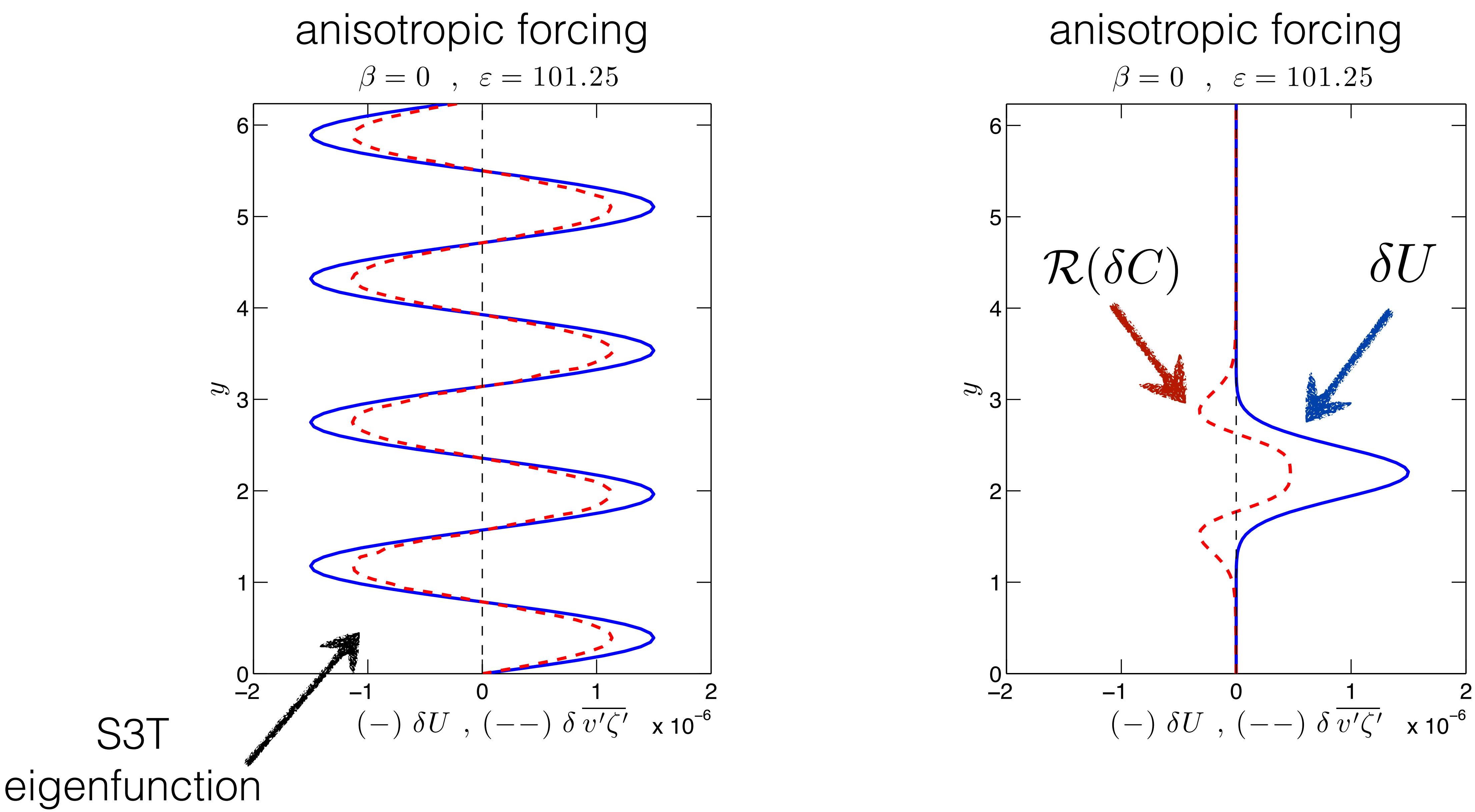
positive contours of f_r in an (n,ϕ) polar plot



dumbbell (?)

(S3T does not include turbulent cascades)

eddies tend to reinforce any mean flow inhomogeneity



turbulence acts anti-diffusively

S3T generalizes the modulational instability of Rossby waves

MI is the hydrodynamic stability of finite amplitude Rossby waves (Lorenz 1972, Gill 1974,Connaughton et al. 2010)

$$\psi_{\mathbf{p}} = A\cos\left(\mathbf{p}\cdot\mathbf{x} - \omega_{\mathbf{p}} t\right)$$

There is a formal equivalence between the modulational instability of $\psi_{\mathbf{p}}$ and the S3T instability of the homogeneous state in the inviscid limit with covariance

$$\hat{C}^e(\mathbf{k}) = (2\pi)^2 p^4 |A|^2 [\delta(\mathbf{k}-\mathbf{p}) + \delta(\mathbf{k}+\mathbf{p})]$$

Parker & Krommes (2015?, *Zonal jets* book) $\longrightarrow$ Same eigenvalue relation

“formal” because the problems are very different

MI: stability of basic state in the form of coherent Rossby wave

S3T: statistical stability of an incoherent state with equilibrium covariance with the same power spectrum as the Rossby wave

S3T generalizes the modulational instability of Rossby waves

The stability of ***any*** coherent nonlinear solution, i.e.,

$$\psi_C = \int_0^{2\pi} \alpha(\theta)\cos[\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t]\,\mathrm{d}\theta$$

$$\mathbf{k} = k_f(\cos\theta, \sin\theta)$$

can be studied as the S3T stability of the homogeneous equilibrium

$$\hat{C}^e(\mathbf{k}) \sim |\alpha(\theta)|^2\,\delta(k - k_f)$$

(which corresponds to the equilibrium covariance in a forced—dissipative flow with forcing structures considered in this talk)

(cf. Bakas, Constantinou & Ioannou 2014)

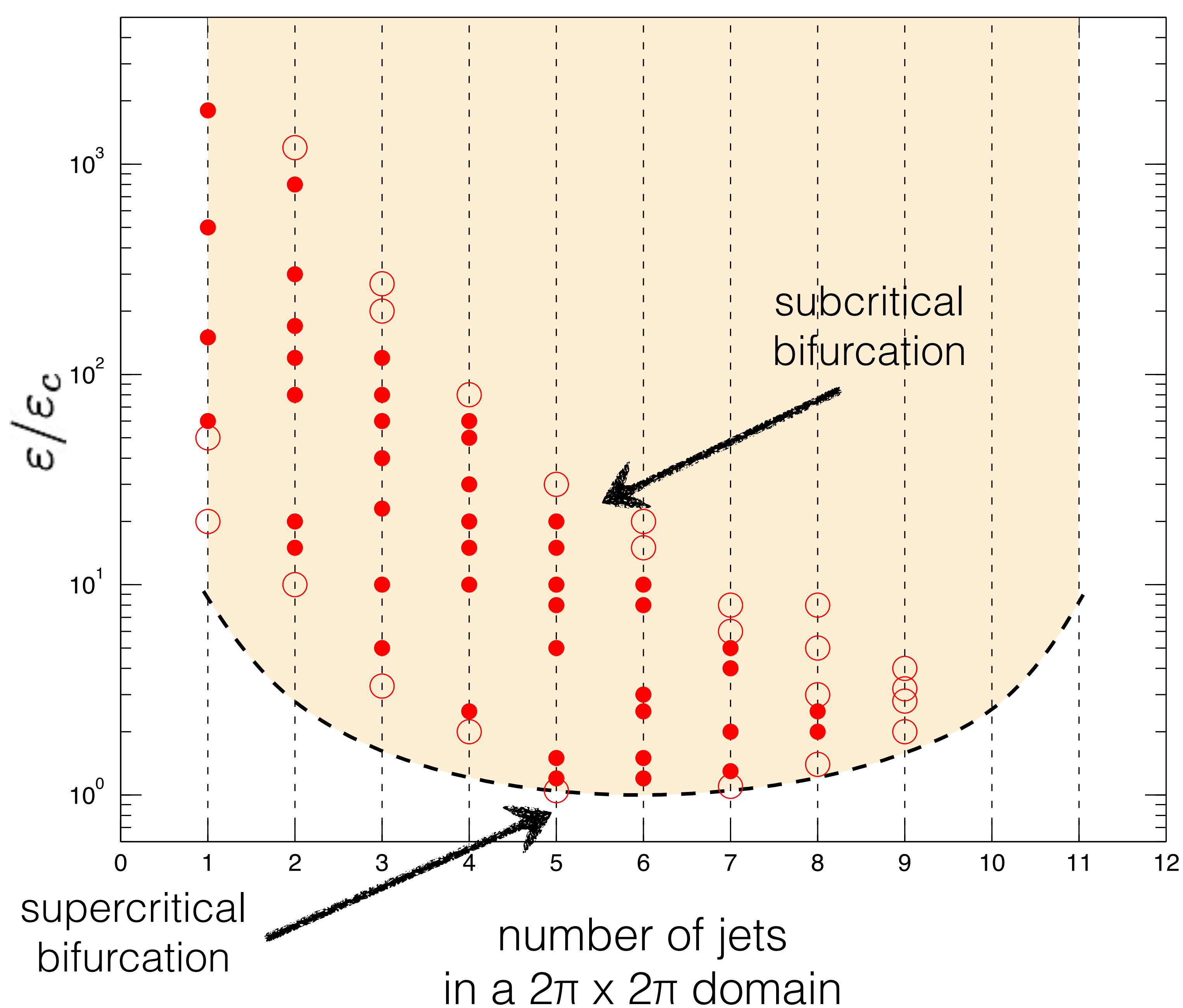
Contrary to modulational instability, which only addresses the homogeneous turbulent equilibrium, within S3T we can also study the stability of any inhomogeneous turbulent equilibrium

Stability of inhomogeneous S3T equilibria

Stability analysis of inhomogeneous turbulence states with zonal jets predicts:

- existence of multiple equilibria and their domain of attraction
- merging of jets as ε increases

For higher energy input rates equilibria become S3T unstable and move towards the left of the diagram



We have discussed the S3T instability of the homogeneous turbulent equilibrium and also the stability of inhomogeneous S3T equilibria characterized by zonal jets.

Are these results reflected in nonlinear simulations?

Nikos showed already a lot of examples in the previous talk.

Also, extensive comparison of the predictions of S3T with nonlinear simulations (bifurcation diagrams, mean flow profiles, jet mergers, etc) can be found in:

Constantinou, Farrell and Ioannou 2014
Bakas and Ioannou 2014

Conclusions

- S3T predicts emergence of jets out of homogeneous turbulence as a bifurcation
- turbulence acts anti-diffusively reinforcing even infinitesimal mean flow inhomogeneities
- S3T stability analysis embeds the modulational instability results into a more general physical framework
- the stability of inhomogeneous statistical turbulent equilibria (i.e. Jupiter) can be studied within S3T framework

References

- Bakas and Ioannou (2013) On the mechanism underlying the spontaneous emergence of barotropic zonal jets. *J. Atmos. Sci.*, **70 (7)**, 2251-2271.
- Constantinou, Farrell & Ioannou (2014) Emergence and equilibration of jets in beta-plane turbulence: applications of Stochastic Structural Stability Theory. *J. Atmos. Sci.*, **71 (5)**, 1818-1842.
- Bakas and Ioannou, (2014) A theory for the emergence of coherent structures in beta-plane turbulence. *J. Fluid Mech.*, **740**, 312-341.
- Bakas, Constantinou and Ioannou (2014) S3T stability of the homogeneous state of barotropic beta-plane turbulence., *J. Atmos. Sci.* (sub judice, arXiv:1407.3354)
- Constantinou, Formation of large-scale structures by turbulence in rotating planets, Ph.D. thesis, (soon!)

thank you!

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