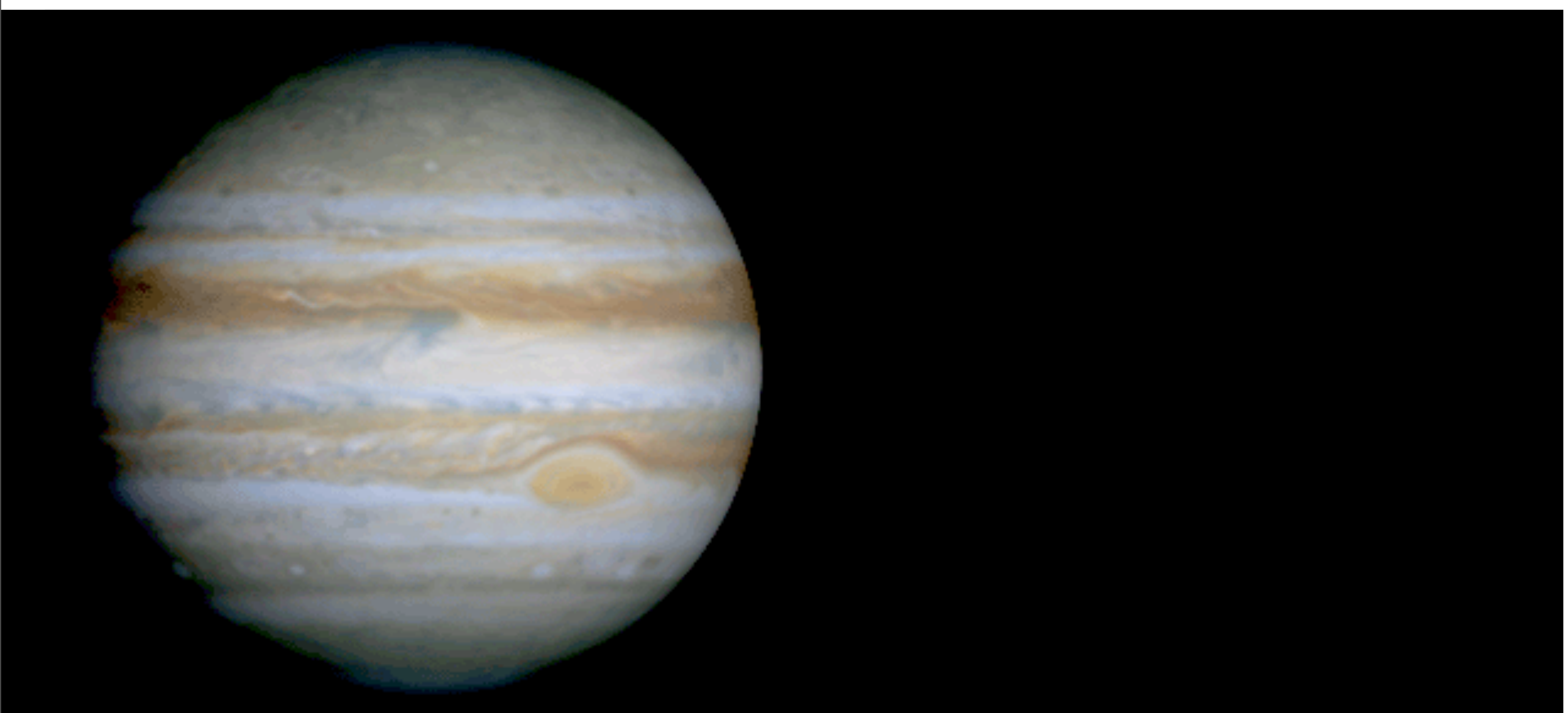
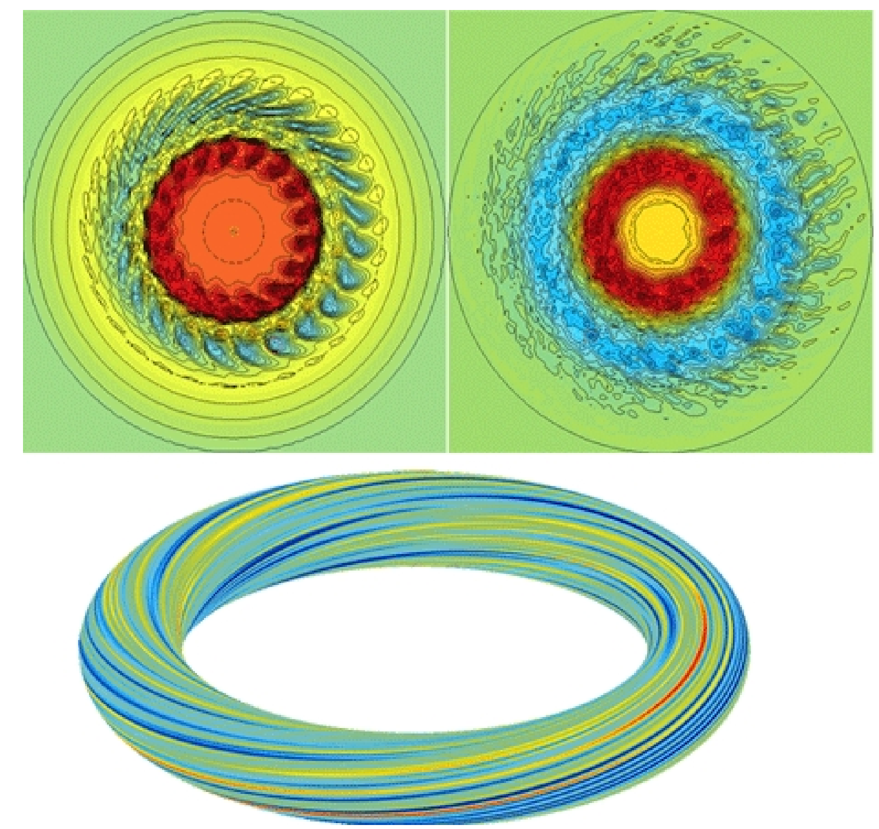
Verification of the predictions of SSST in nonlinear simulations

Navid Constantinou
Physics Department
National and Kapodistrian University of Athens

ISSI Group Meeting
April 2013

zonal flows coexist with turbulence





banded Jovian jets

NASA/Cassini Jupiter Images

polar front jet

NASA/Goddard Space Flight Center

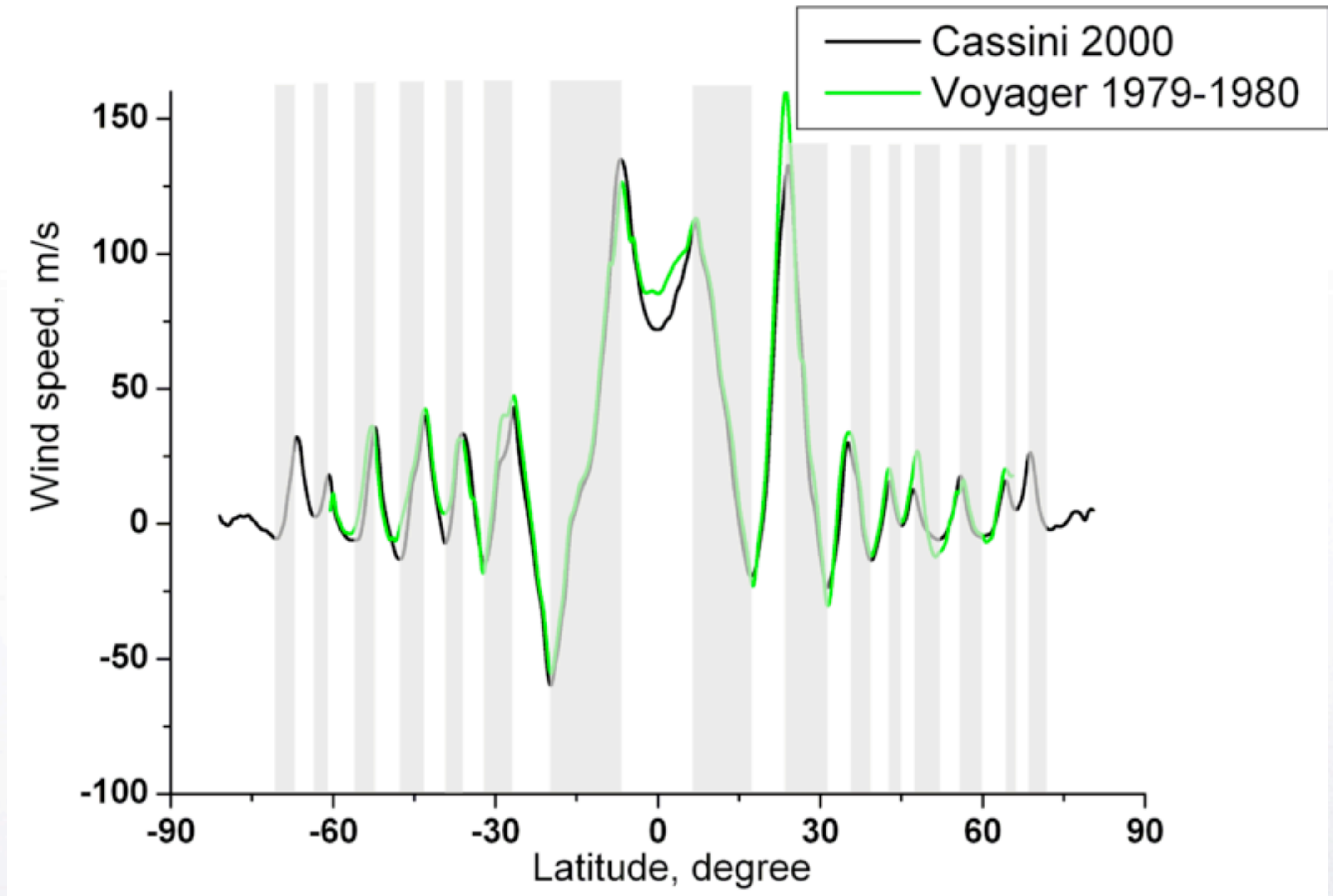
jets in tokamaks

courtesy: L. Villard

zonal flows coexist with turbulence

banded Jovian jets

NASA/Cassini Jupiter Images

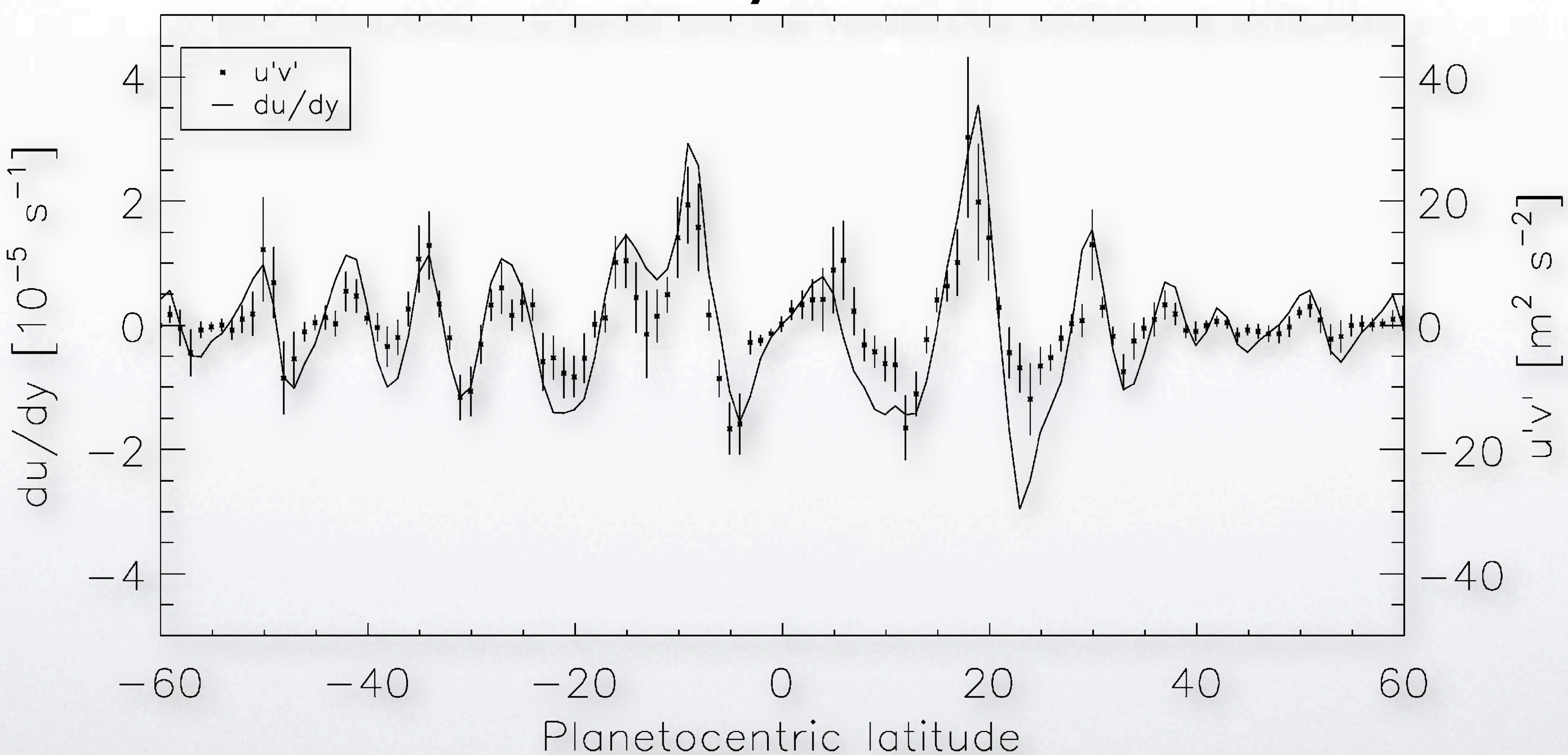


observed Jovian zonal winds at cloud level

Vasadava & Showman, 2005

zonal flows are maintained by eddies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{U^2}{2}\,\mathrm{d}y = \int \frac{\mathrm{d}U}{\mathrm{d}y}\overline{u'v'}\,\mathrm{d}y - \text{Dissipation}$$



(Salyk et. al. 2006)

Zonal - Eddy field decomposition

$$\varphi(x,y,t) = \Phi(y,t) + \varphi'(x,y,t)$$

where $$\Phi(y,t) = \overline{\varphi}(y,t) = \frac{1}{L_x}\int_0^{L_x} \varphi(x',y,t)\,\mathrm{d}x'$$

Zonal - Eddy field decomposition

$$\varphi(x,y,t) = \underbrace{\Phi(y,t)}_{\text{zonal mean}} + \underbrace{\varphi'(x,y,t)}_{\text{eddy}}$$

where $$\Phi(y,t) = \overline{\varphi}(y,t) = \frac{1}{L_x}\int_0^{L_x} \varphi(x',y,t)\,\mathrm{d}x'$$

SSST System

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \sum_{k=1}^{N_k} \frac{k}{2} \,\mathrm{Im}\left[\mathrm{vecd}\left(\boldsymbol{\Delta}_k^{-1}\,\mathbf{C}_k\right)\right] - r_{\mathrm{m}} U$$

$$\frac{\mathrm{d}\mathbf{C}_k}{\mathrm{d}t} = \mathbf{A}_k(U)\,\mathbf{C}_k + \mathbf{C}_k\,\mathbf{A}_k(U)^\dagger + \epsilon\,\mathbf{Q}_k$$

admits equilibria $(U^E, \mathbf{C}_1^E, \dots, \mathbf{C}_{N_k}^E) \equiv (U^E, \mathbf{C}_k^E)$

homogeneous equilibrium: $(U^E = 0, \mathbf{C}_k^E = \epsilon \mathbf{Q}_k^E / 2r)$

SSST System

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \sum_{k=1}^{N_k} \frac{k}{2} \,\mathrm{Im}\left[\mathrm{vecd}\left(\mathbf{\Delta}_k^{-1}\,\mathbf{C}_k\right)\right] - r_m U$$

$$\overline{v'\zeta'}$$

$$\frac{\mathrm{d}\mathbf{C}_k}{\mathrm{d}t} = \mathbf{A}_k(U)\,\mathbf{C}_k + \mathbf{C}_k\,\mathbf{A}_k(U)^\dagger + \epsilon\,\mathbf{Q}_k$$

admits equilibria $(U^E, \mathbf{C}_1^E, \dots, \mathbf{C}_{N_k}^E) \equiv (U^E, \mathbf{C}_k^E)$

homogeneous equilibrium: $(U^E = 0, \mathbf{C}_k^E = \epsilon \mathbf{Q}_k^E / 2r)$

SSST System

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \sum_{k=1}^{N_k} \frac{k}{2} \mathrm{Im}\left[\mathrm{vecd}\left(\boldsymbol{\Delta}_k^{-1}\,\mathbf{C}_k\right)\right] - r_{\mathrm{m}} U$$

$$\frac{\mathrm{d}\mathbf{C}_k}{\mathrm{d}t} = \mathbf{A}_k(U)\,\mathbf{C}_k + \mathbf{C}_k\,\mathbf{A}_k(U)^\dagger + \epsilon\,\mathbf{Q}_k$$

admits equilibria $(U^E, \mathbf{C}_1^E, \ldots, \mathbf{C}_{N_k}^E) \equiv (U^E, \mathbf{C}_k^E)$

homogeneous equilibrium: $(U^E = 0, \mathbf{C}_k^E = \epsilon \mathbf{Q}_k^E / 2r)$

SSST System

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \sum_{k=1}^{N_k} \frac{k}{2}\,\mathrm{Im}\left[\mathrm{vecd}\left(\boldsymbol{\Delta}_k^{-1}\,\mathbf{C}_k\right)\right] - r_{\mathrm{m}}U$$

$$\frac{\mathrm{d}\mathbf{C}_k}{\mathrm{d}t} = \mathbf{A}_k(U)\,\mathbf{C}_k + \mathbf{C}_k\,\mathbf{A}_k(U)^\dagger + \epsilon\,\mathbf{Q}_k$$

admits equilibria $(U^E, \mathbf{C}^E$

homogeneous equilibrium

N_y discretization points and N_k zonal harmonics $\Rightarrow$
$\Rightarrow$ $(U, \mathbf{C}_k)$ is a $(2N_kN_y^2 + N_y)$ state vector

(e.g. $Ny = 128, N_k = 10 \;\Rightarrow\; (U, \mathbf{C}_k) \sim 3.3 \times 10^5$)

SSST System

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \sum_{k=1}^{N_k} \frac{k}{2} \operatorname{Im}\left[\operatorname{vecd}\left(\mathbf{\Delta}_k^{-1}\,\mathbf{C}_k\right)\right] - r_{\mathrm{m}} U$$

$$\frac{\mathrm{d}\mathbf{C}_k}{\mathrm{d}t} = \mathbf{A}_k(U)\,\mathbf{C}_k + \mathbf{C}_k\,\mathbf{A}_k(U)^{\dagger} + \epsilon\,\mathbf{Q}_k$$

admits equilibria $(U^E, \mathbf{C}_1^E, \ldots, \mathbf{C}_{N_k}^E) \equiv (U^E, \mathbf{C}_k^E)$

homogeneous equilibrium: $(U^E = 0, \mathbf{C}_k^E = \epsilon \mathbf{Q}_k^E / 2r)$

Stability of SSST equilibrium

perturbing the SSST equilibrium $(U^E + \delta U, \mathbf{C}_k^E + \delta \mathbf{C}_k)$

$$\frac{\mathrm{d}\delta U}{\mathrm{d}t} = \sum_{k=1}^{N_k} \frac{k}{2} \,\mathrm{Im}\left[\mathrm{vecd}\left(\boldsymbol{\Delta}_k^{-1}\,\delta\mathbf{C}_k\right)\right] - r_\mathrm{m}\,\delta U$$

$$\frac{\mathrm{d}\,\delta\mathbf{C}_k}{\mathrm{d}t} = \mathbf{A}_k(U^E)\,\delta\mathbf{C}_k + \delta\mathbf{C}_k\,\mathbf{A}_k(U^E)^\dagger + \delta\mathbf{A}_k\,\mathbf{C}_k^E + \mathbf{C}_k^E\,\delta\mathbf{A}_k^\dagger$$

with $\delta\mathbf{A}_k \equiv \mathbf{A}_k(U^E + \delta U) - \mathbf{A}_k(U^E)$

Stability of SSST equilibrium

perturbing the SSST equilibrium $\quad (U^E + \delta U, \mathbf{C}_k^E + \delta \mathbf{C}_k)$

$$\frac{\mathrm{d}\delta U}{\mathrm{d}t} = \sum_{k=1}^{N_k} \frac{k}{2}\,\mathrm{Im}\left[\mathrm{vecd}\left(\boldsymbol{\Delta}_k^{-1}\,\delta\mathbf{C}_k\right)\right] - r_\mathrm{m}\,\delta U$$

$$\frac{\mathrm{d}\,\delta\mathbf{C}_k}{\mathrm{d}t} = \mathbf{A}_k(U^E)\,\delta\mathbf{C}_k + \delta\mathbf{C}_k\,\mathbf{A}_k(U^E)^\dagger + \delta\mathbf{A}_k\,\mathbf{C}_k^E + \mathbf{C}_k^E\,\delta\mathbf{A}_k^\dagger$$

$$\text{with } \delta\mathbf{A}_k \equiv \mathbf{A}_k(U^E + \delta U) - \mathbf{A}_k(U^E)$$

split for real/imaginary part and
search for eigensolutions: $(\delta\hat{U}, \delta\hat{\mathbf{C}}_{k,R}, \delta\hat{\mathbf{C}}_{k,I})\,\mathrm{e}^{\sigma t} \quad$ for $k = 1, \dots, N_k$,

$$\sigma \begin{pmatrix} \delta\hat{U} \\ \delta\hat{\mathbf{C}}_{k,R} \\ \delta\hat{\mathbf{C}}_{k,I} \end{pmatrix} = \mathbb{L} \begin{pmatrix} \delta\hat{U} \\ \delta\hat{\mathbf{C}}_{k,R} \\ \delta\hat{\mathbf{C}}_{k,I} \end{pmatrix}$$

Stability of SSST equilibrium

perturbing the SSST equilibrium $(U^E + \delta U, \mathbf{C}_k^E + \delta\mathbf{C}_k)$

$$\frac{\mathrm{d}\delta U}{\mathrm{d}t} = \sum_{k=1}^{N_k} \frac{k}{2}\,\mathrm{Im}\left[\mathrm{vecd}\left(\boldsymbol{\Delta}_k^{-1}\,\delta\mathbf{C}_k\right)\right] - r_\mathrm{m}\,\delta U$$

$$\frac{\mathrm{d}\,\delta\mathbf{C}_k}{\mathrm{d}t} = \mathbf{A}_k(U^E)\,\delta\mathbf{C}_k + \delta\mathbf{C}_k\,\mathbf{A}_k(U^E)^\dagger + \delta\mathbf{A}_k\,\mathbf{C}_k^E + \mathbf{C}_k^E\,\delta\mathbf{A}_k^\dagger$$

with $\delta\mathbf{A}_k \equiv \mathbf{A}_k(U^E + \delta U) - \mathbf{A}_k(U^E)$

split for real/imaginary part and
search for eigensolutions: $(\delta\hat{U}, \delta\hat{\mathbf{C}}_{k,R}, \delta\hat{\mathbf{C}}_{k,I})\,\mathrm{e}^{\sigma t}$ for $k = 1, \dots, N_k$,

$$\sigma \begin{pmatrix} \delta\hat{U} \\ \delta\hat{\mathbf{C}}_{k,R} \\ \delta\hat{\mathbf{C}}_{k,I} \end{pmatrix} = \mathbb{L} \begin{pmatrix} \delta\hat{U} \\ \delta\hat{\mathbf{C}}_{k,R} \\ \delta\hat{\mathbf{C}}_{k,I} \end{pmatrix}$$

$\mathbb{L} = \mathbb{L}(U^E, \mathbf{C}_k^E)$ has dimension $(2N_k N_y^2 + N_y)$

Eigenanalysis of stability operator

for $(U^E = 0, \mathbf{C}_k^E = \epsilon \mathbf{Q}_k^E / 2r) \;\Rightarrow\; \delta\hat{U} = e^{iny}$

1. assume $\sigma, \delta\hat{U}_n \;\Rightarrow\; \delta\hat{\mathbf{C}}_{k,n} \;\Rightarrow\; \delta(\overline{v'\zeta'})_n$
2. use mean flow perturbation equation

$$\sigma\delta\hat{U}_n = \delta(\overline{v'\zeta'})_n - r_\mathrm{m}\delta\hat{U}_n$$

to calculate new σ

3. repeat until convergence

(this method can be generalized for $U^E \neq 0$ equilibria)

Eigenanalysis of stability operator

for $(U^E = 0, \mathbf{C}_k^E = \epsilon \mathbf{Q}_k^E/2r) \;\Rightarrow\; \delta\hat{U} = e^{iny}$

1. assume $\sigma, \delta\hat{U}_n \;\Rightarrow\; \delta\hat{\mathbf{C}}_{k,n} \;\Rightarrow\; \delta(\overline{v'\zeta'})_n$
2. use mean flow perturbation equation
$$\sigma\delta\hat{U}_n = \delta(\overline{v'\zeta'})_n - r_{\mathrm{m}}\delta\hat{U}_n$$
to calculate new σ
3. repeat until convergence

this method works extremely well and allows us to attack the $(2N_kN_y^2+N_y)$x $(2N_kN_y^2+N_y)$ eigenvalue problem which would otherwise be impossible

(this method can be generalized for $U^E \neq 0$ equilibria)

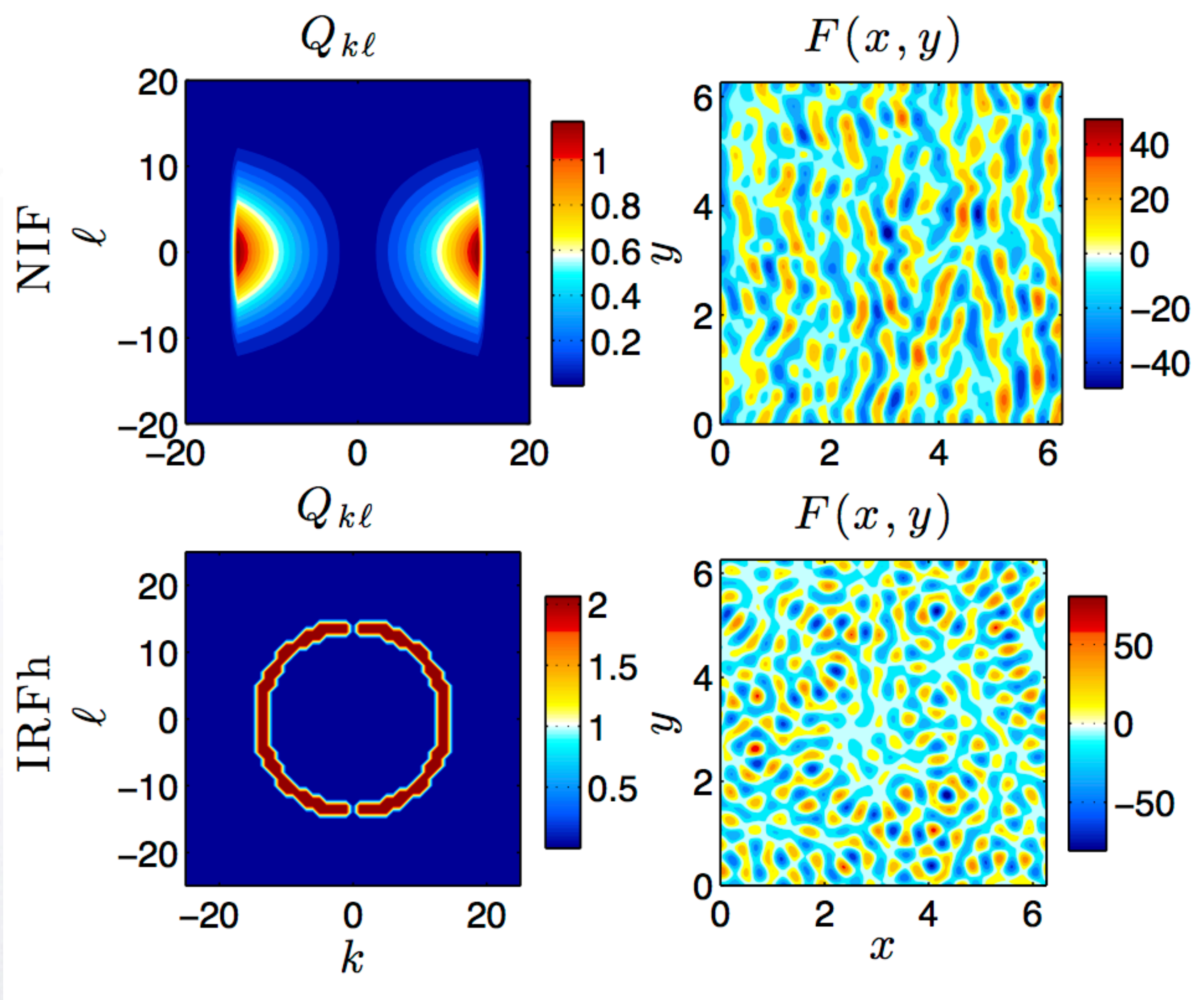
Types of stochastic forcing: NIF & IRFh

Non-isotropic Forcing (NIF)

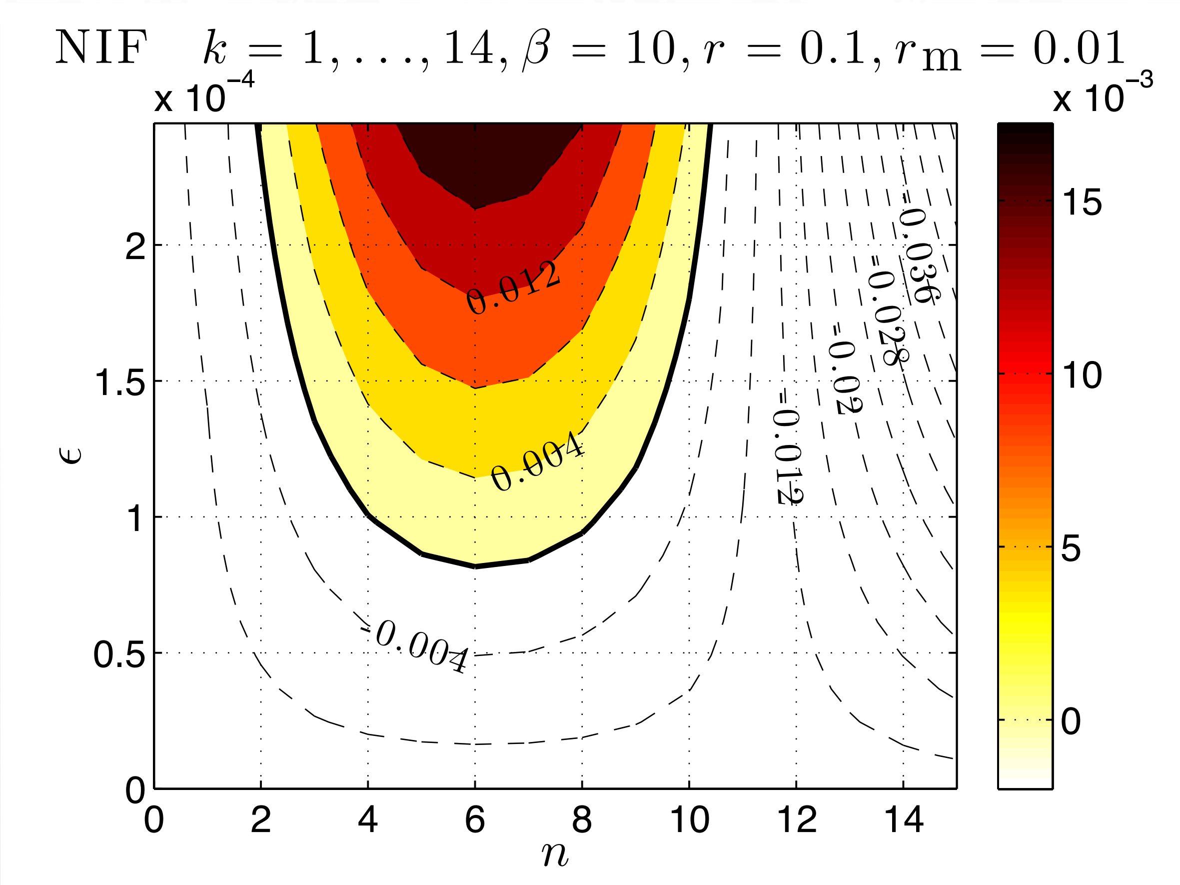
parametrize excitation by baroclinic instabilities

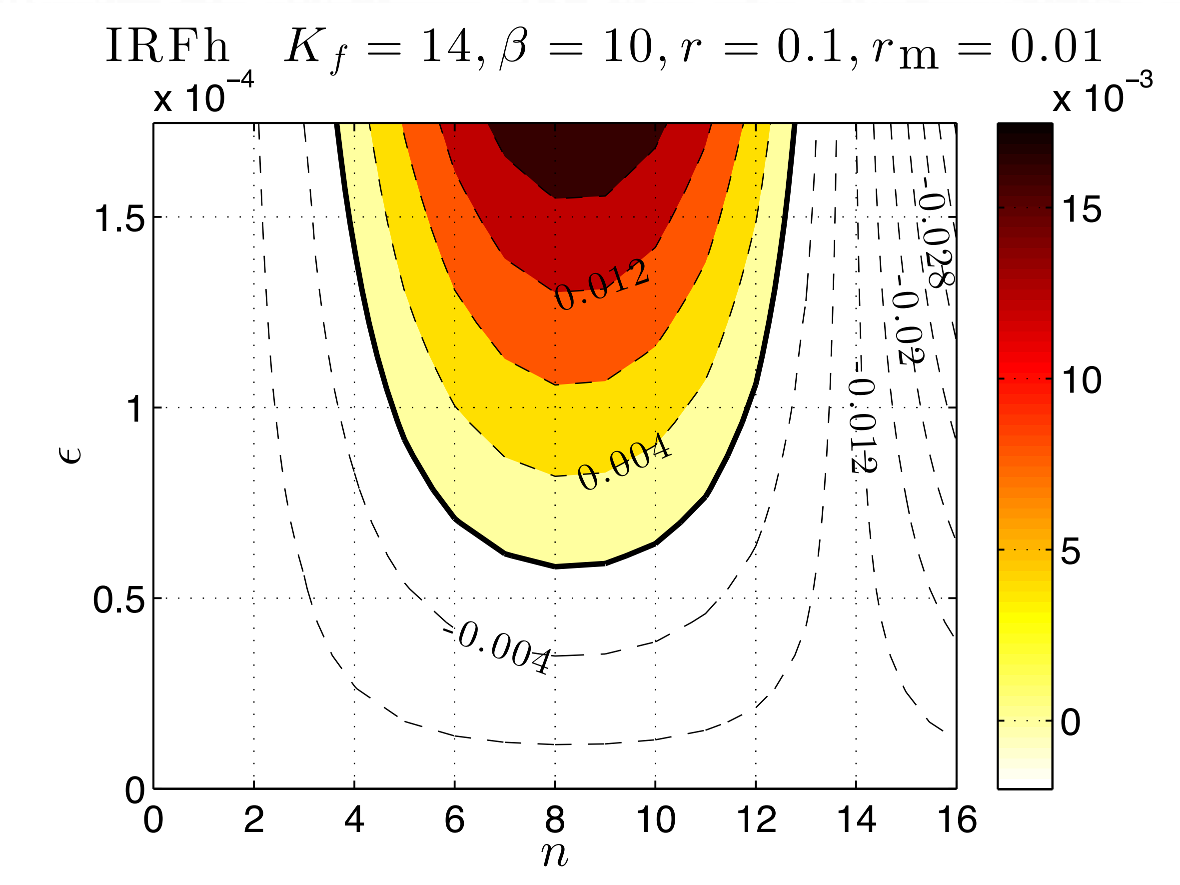
Isotropic Ring Forcing (IRFh)

parametrize excitation by convection

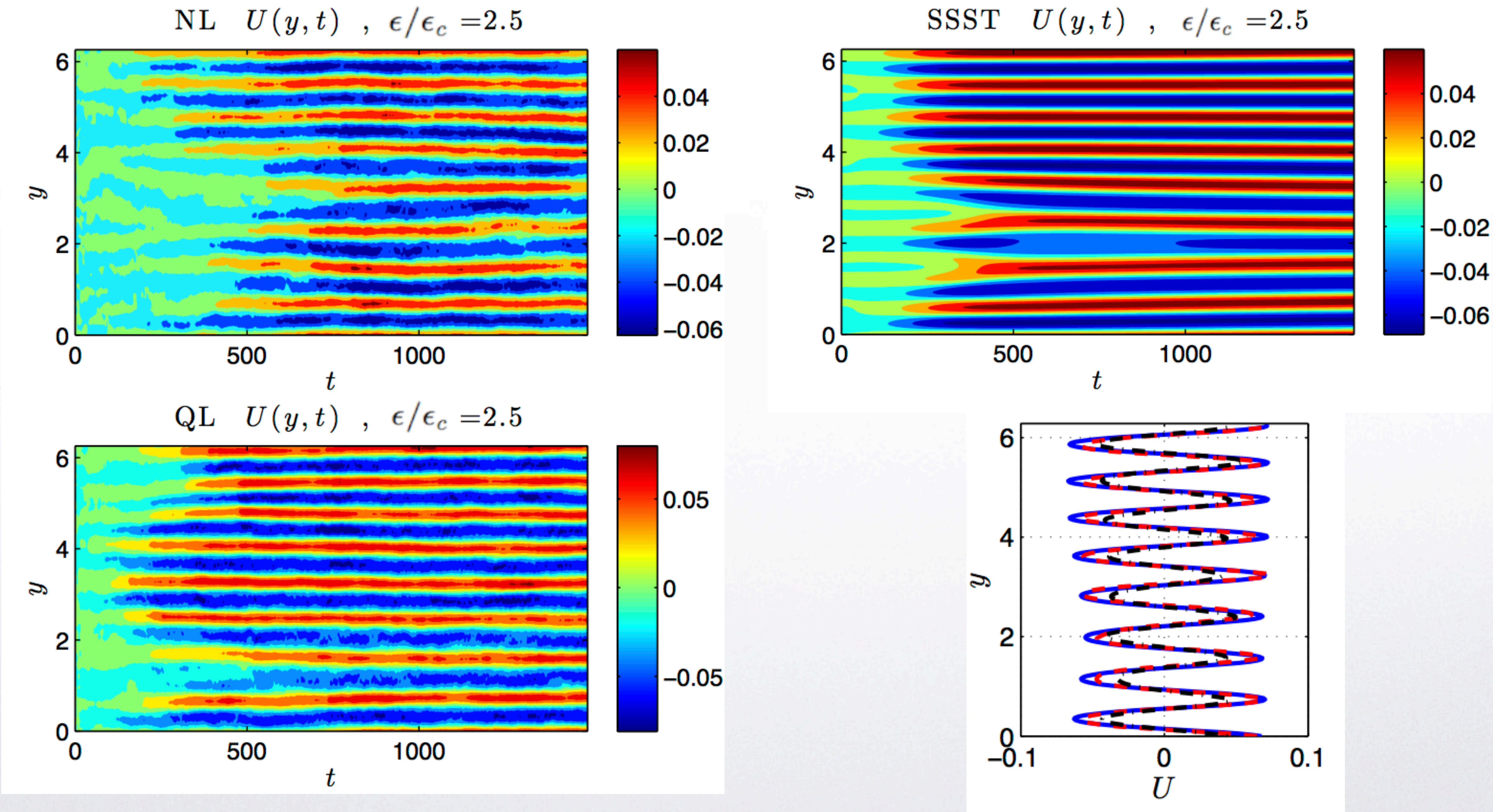


SSST growth rates for homogeneous equilibrium

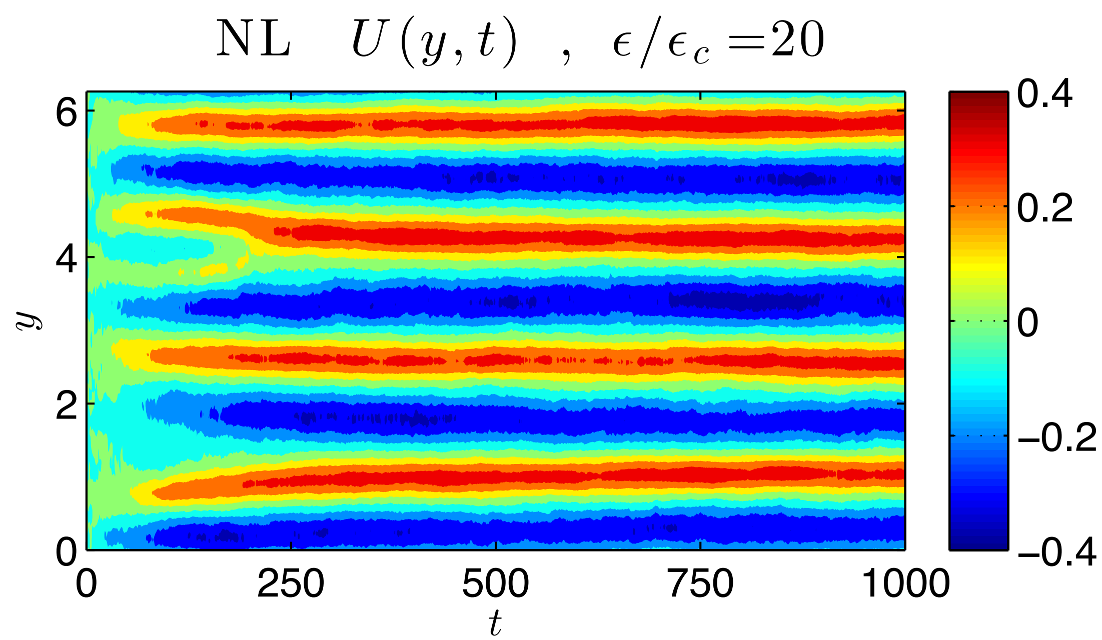


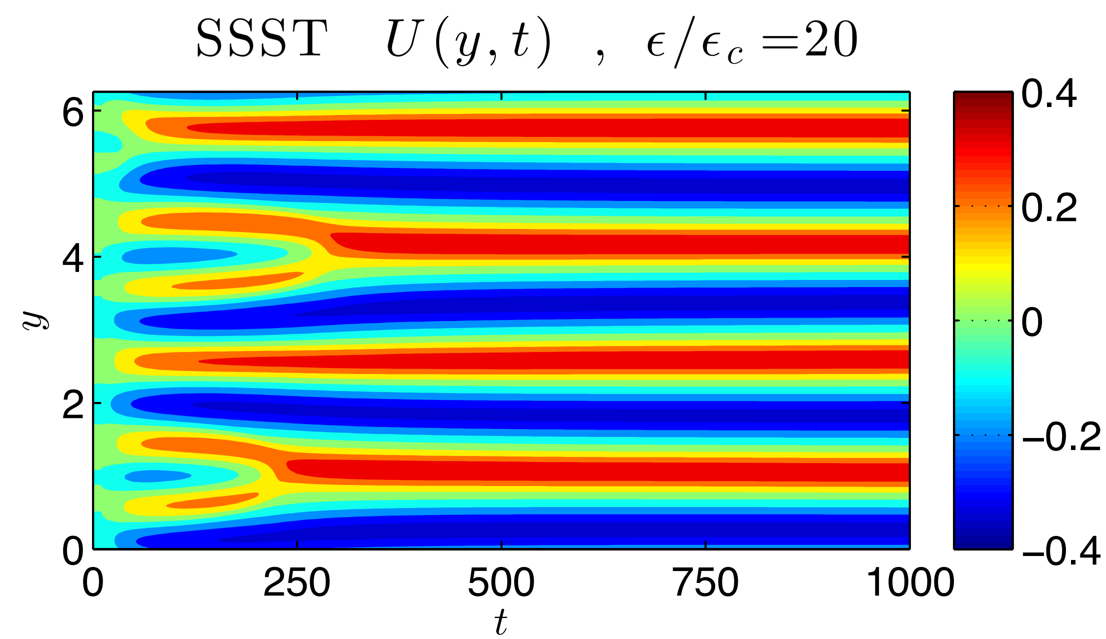


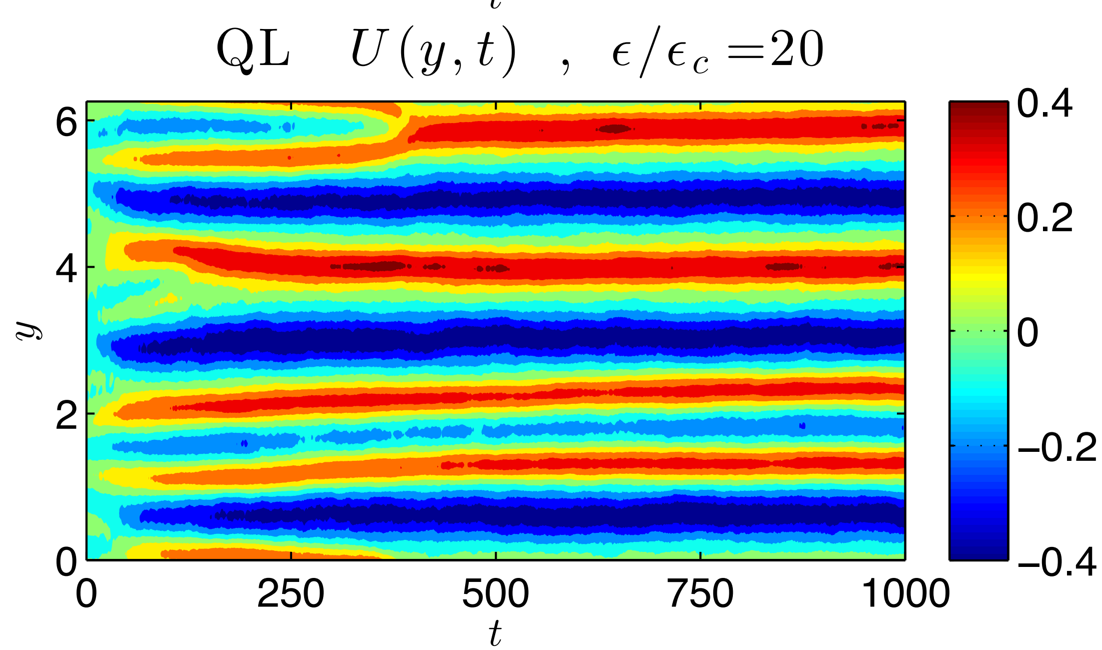
IRFh forcing, $r_{\mathrm{m}} = r/10$ (NL, QL, SSST)

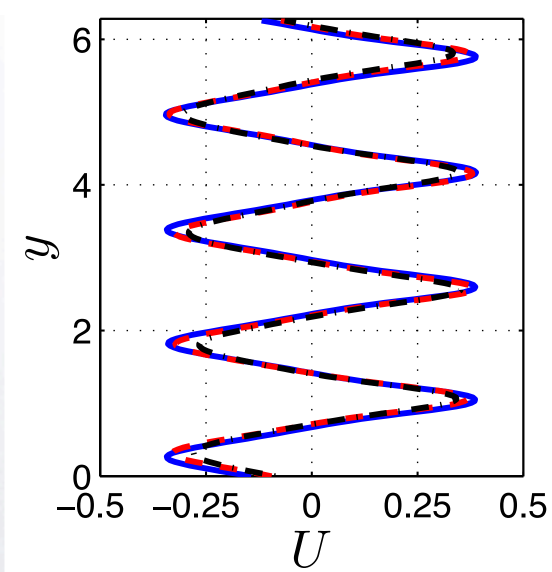


NIF forcing, $r_m = r/10$ (NL, QL, SSST)



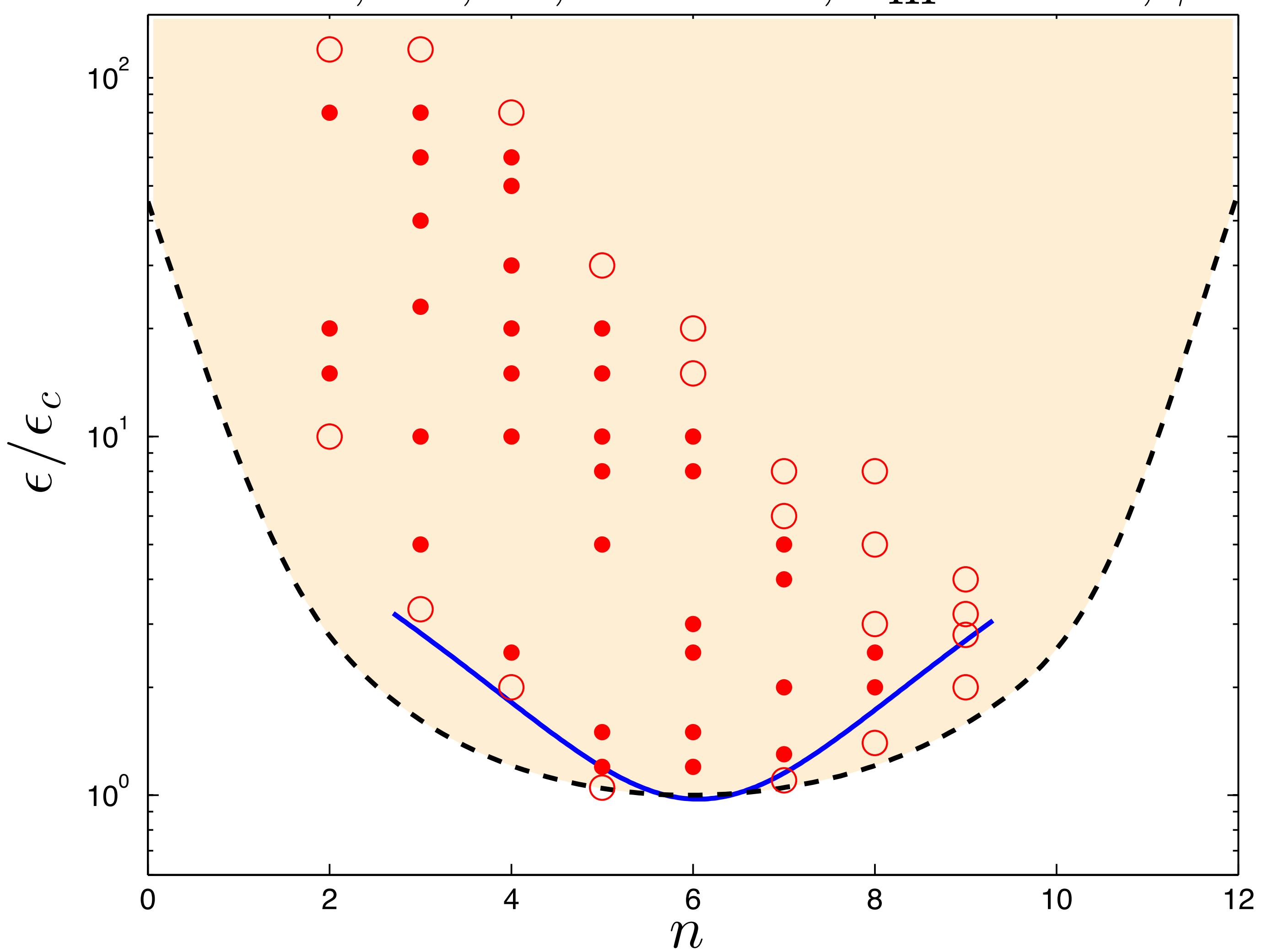






SSST equilibria stability diagram

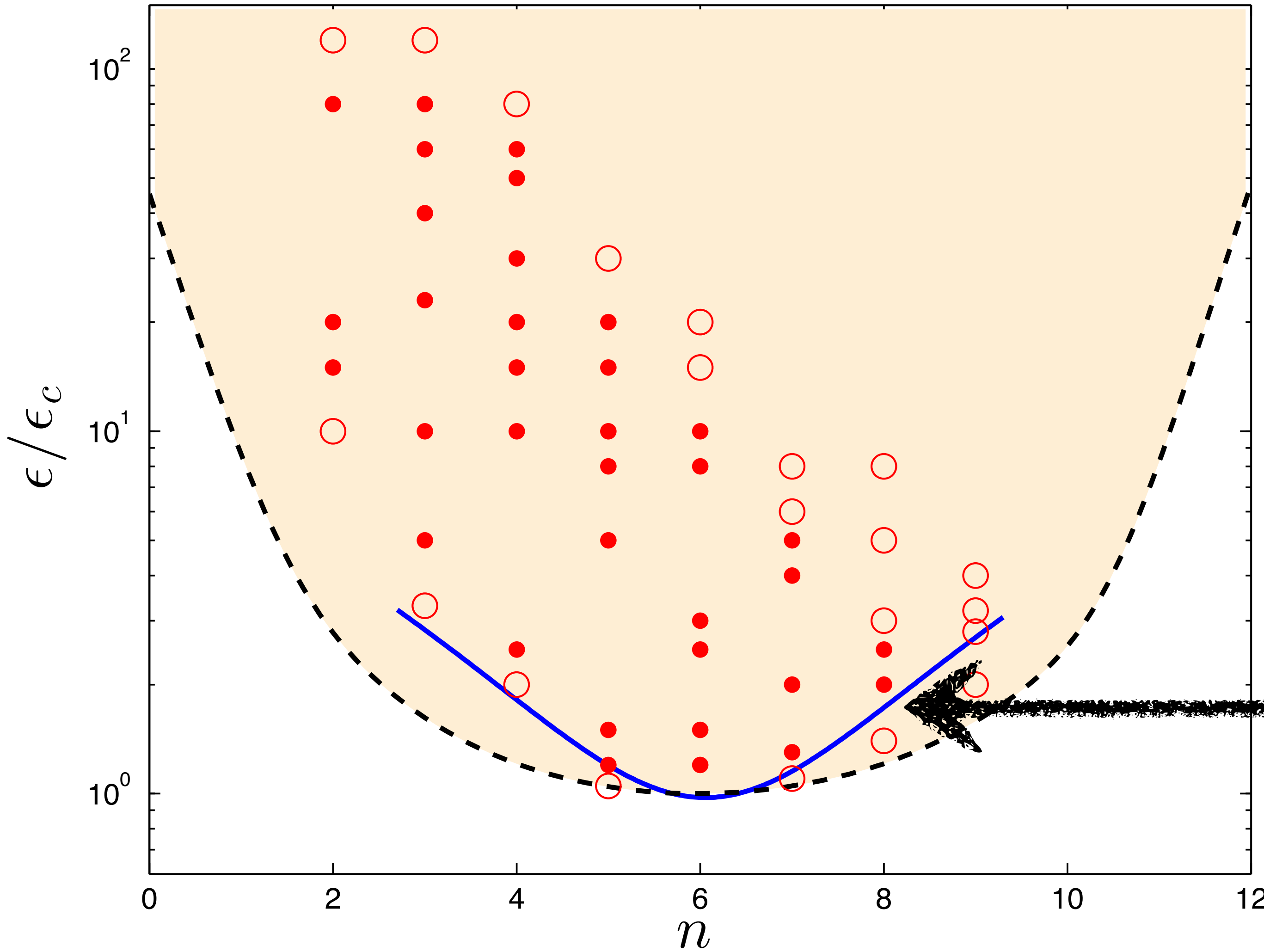
NIF at $k = 1, \ldots, 14$, $r = 10^{-1}$, $r_{\mathrm{m}} = 10^{-2}$, $\beta = 10$



For higher energy input rates equilibria become SSST unstable and move towards the left of the diagram

SSST equilibria stability diagram

NIF at $k = 1, \ldots, 14$, $r = 10^{-1}$, $r_{\mathrm{m}} = 10^{-2}$, $\beta = 10$

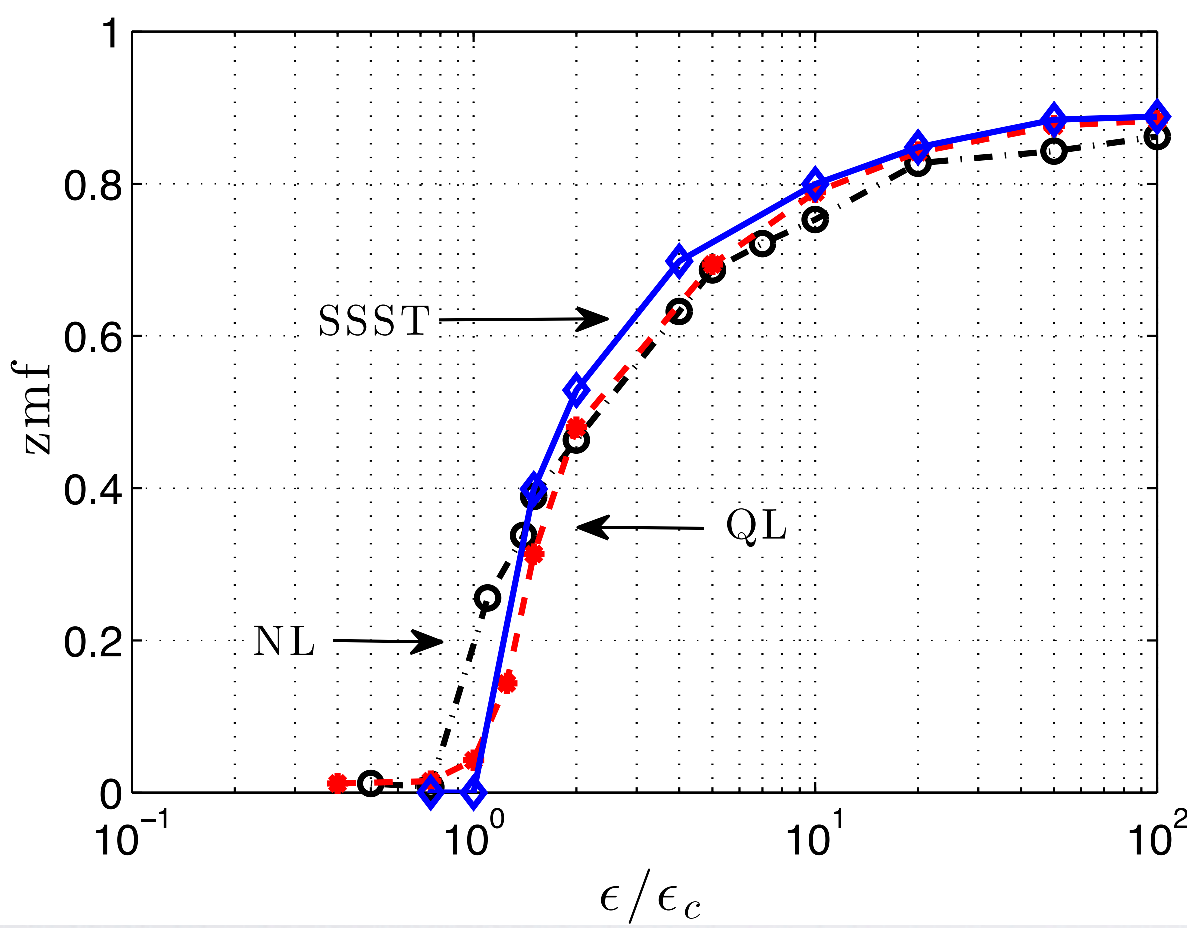


For higher energy input rates equilibria become SSST unstable and move towards the left of the diagram

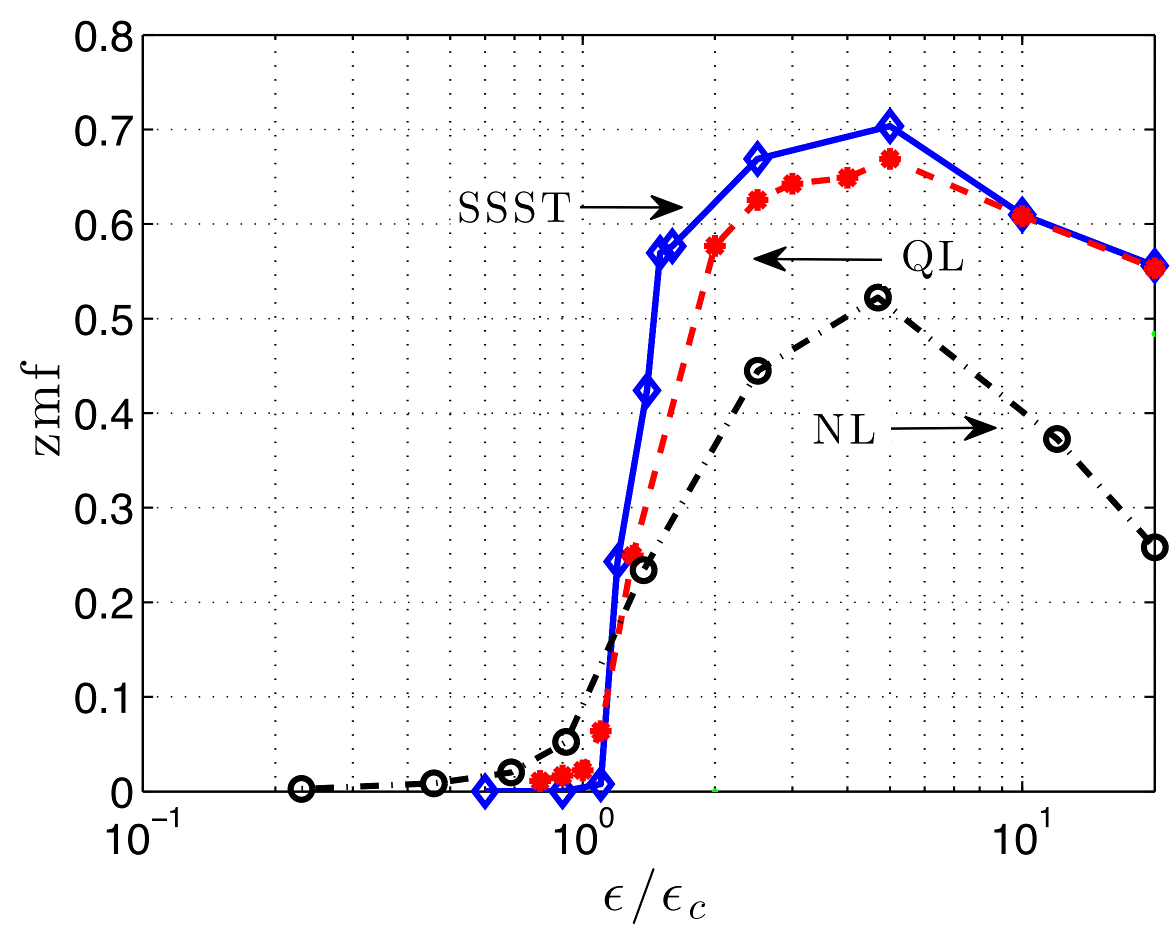
Eckhaus instability boundary

Bifurcation diagrams

NIF, $r = 0.1$, $r_{\mathrm{m}} = r/10$

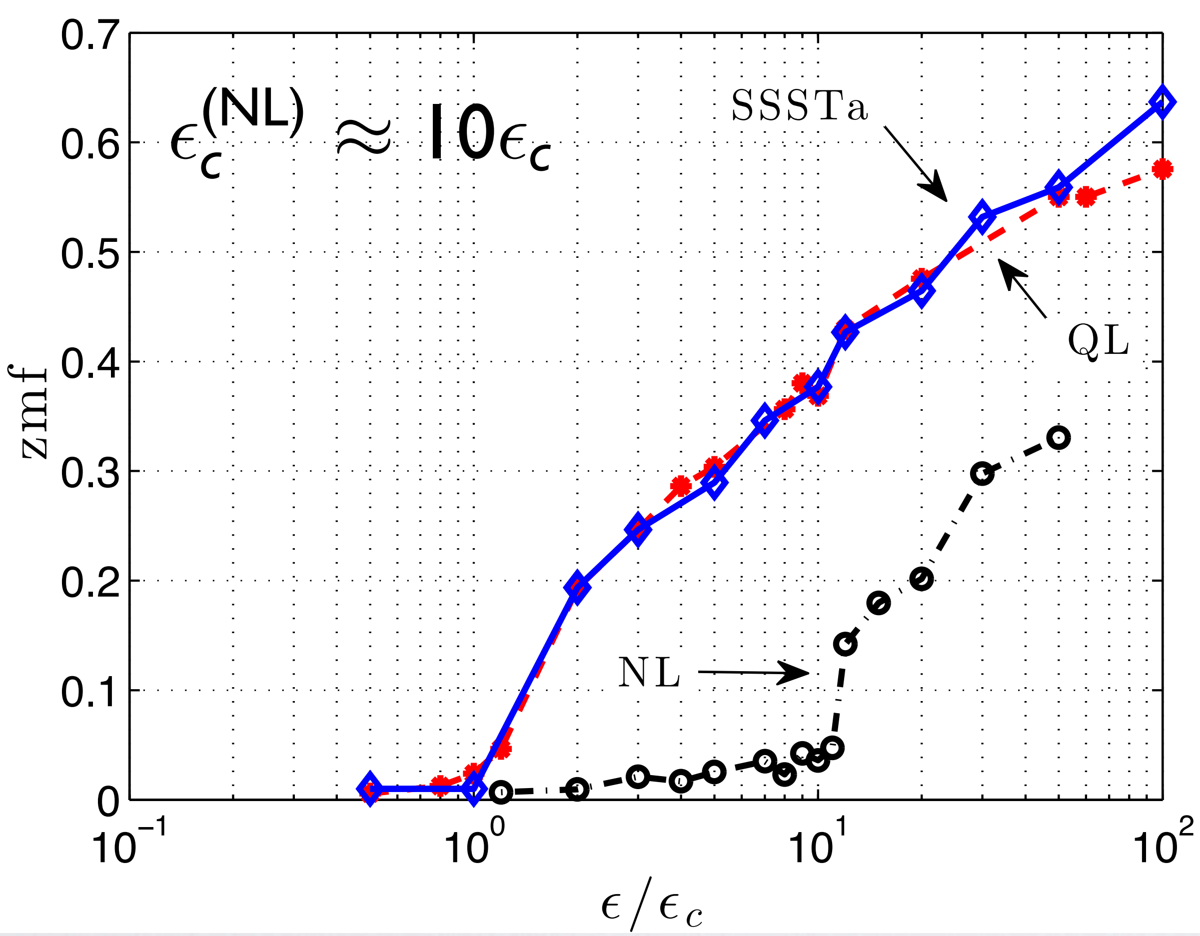


IRFh, $r = 0.1$, $r_{\mathrm{m}} = r/10$

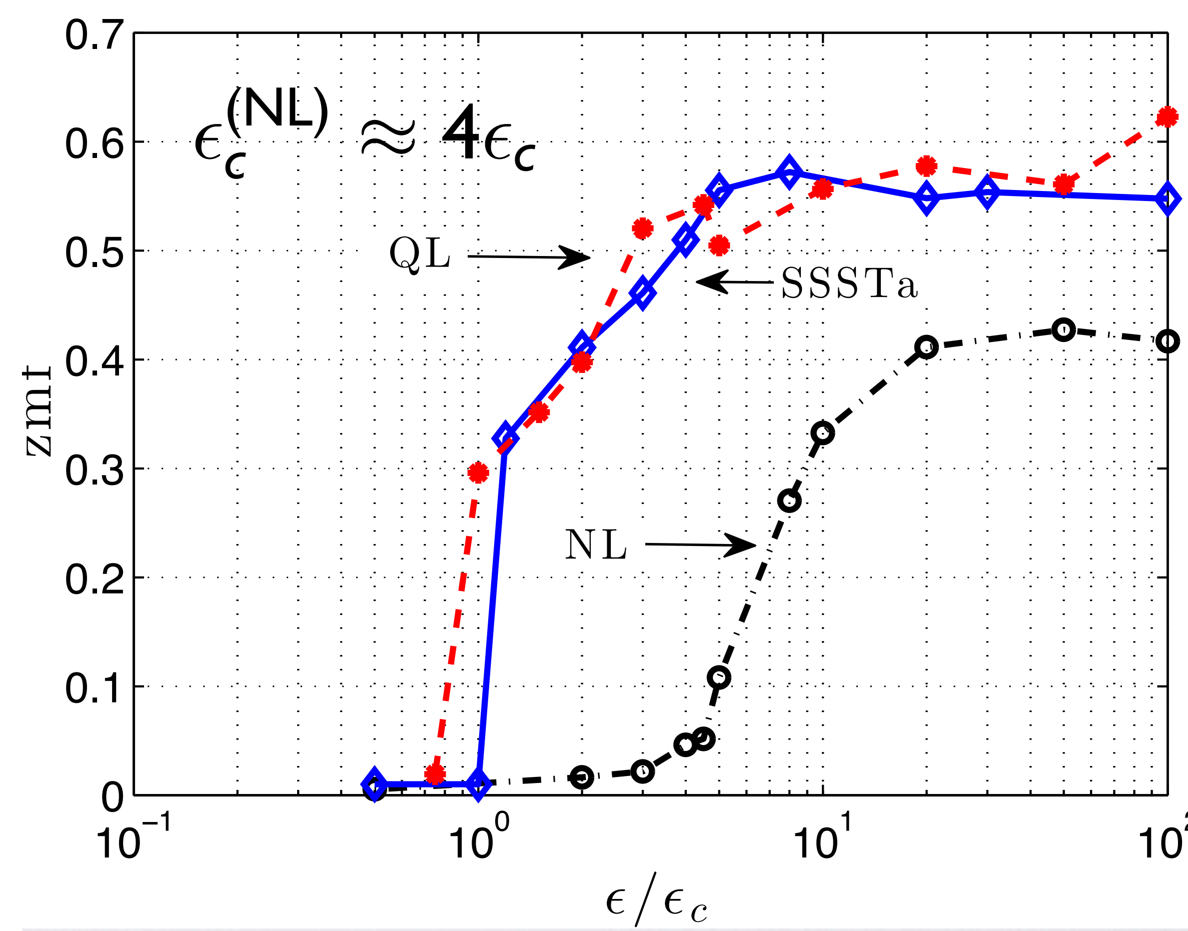


Bifurcation diagrams

NIF, $r = r_{\mathrm{m}} = 10^{-2}$



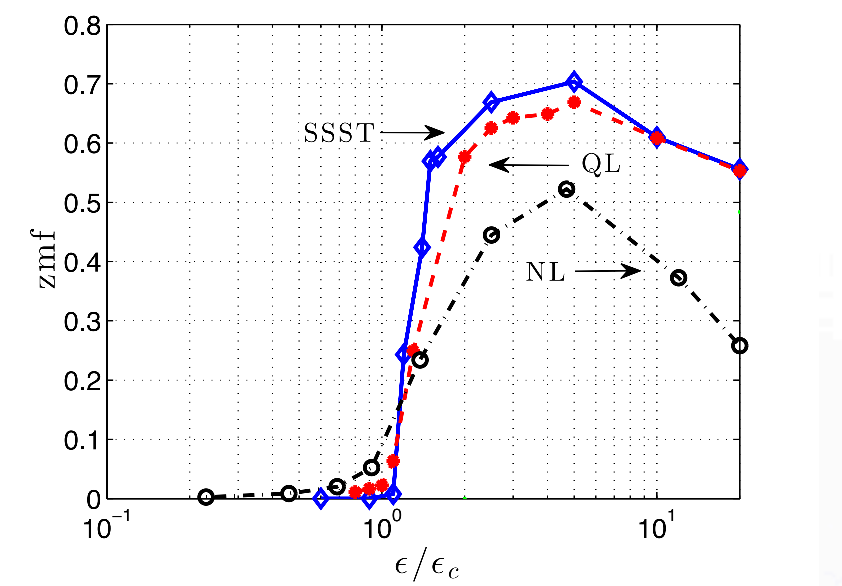
IRFh, $r = r_{\mathrm{m}} = 10^{-2}$



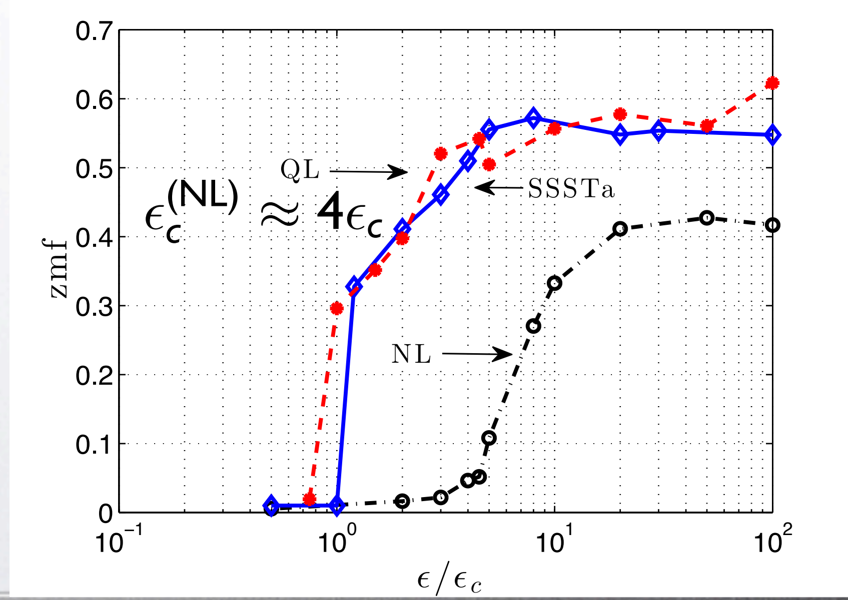
Issues to be clarified

- What happens below the critical energy input rate, ϵ_c ?
- When does the predicted ϵ_c agrees with the NL bifurcation point, $\epsilon_c^{(NL)}$ and why they don't always agree?
- What happens in the range $\epsilon_c < \epsilon < \epsilon_c^{(NL)}$?
- How can $\epsilon_c^{(NL)}$ be predicted?

IRFh, $r = 0.1$, $r_{\mathrm{m}} = r/10$

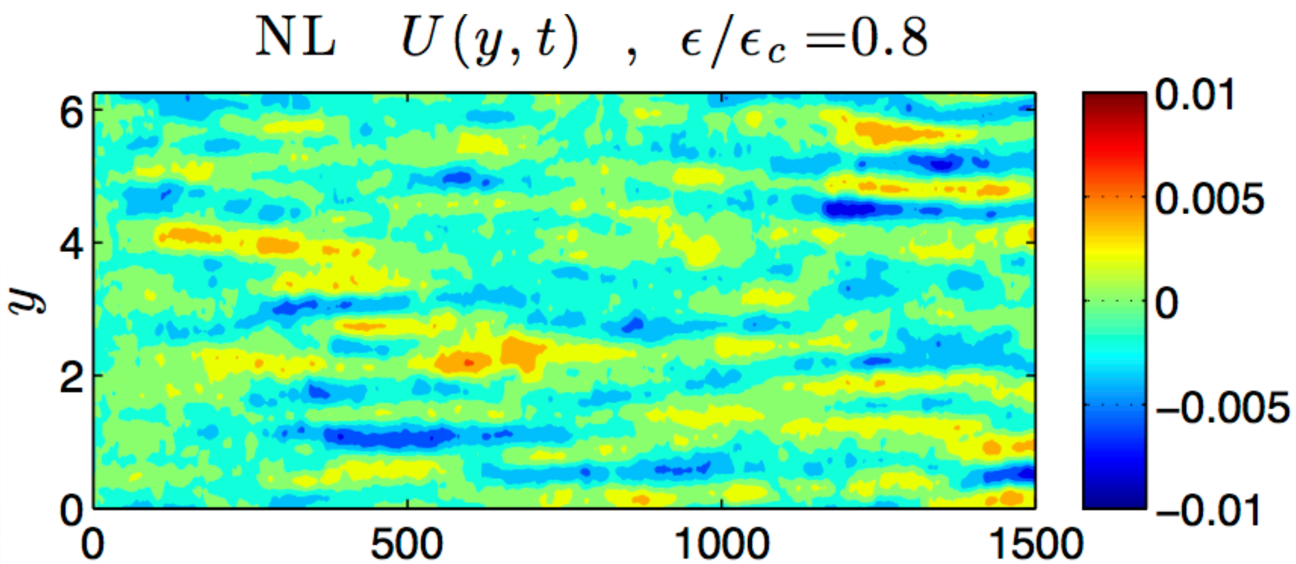


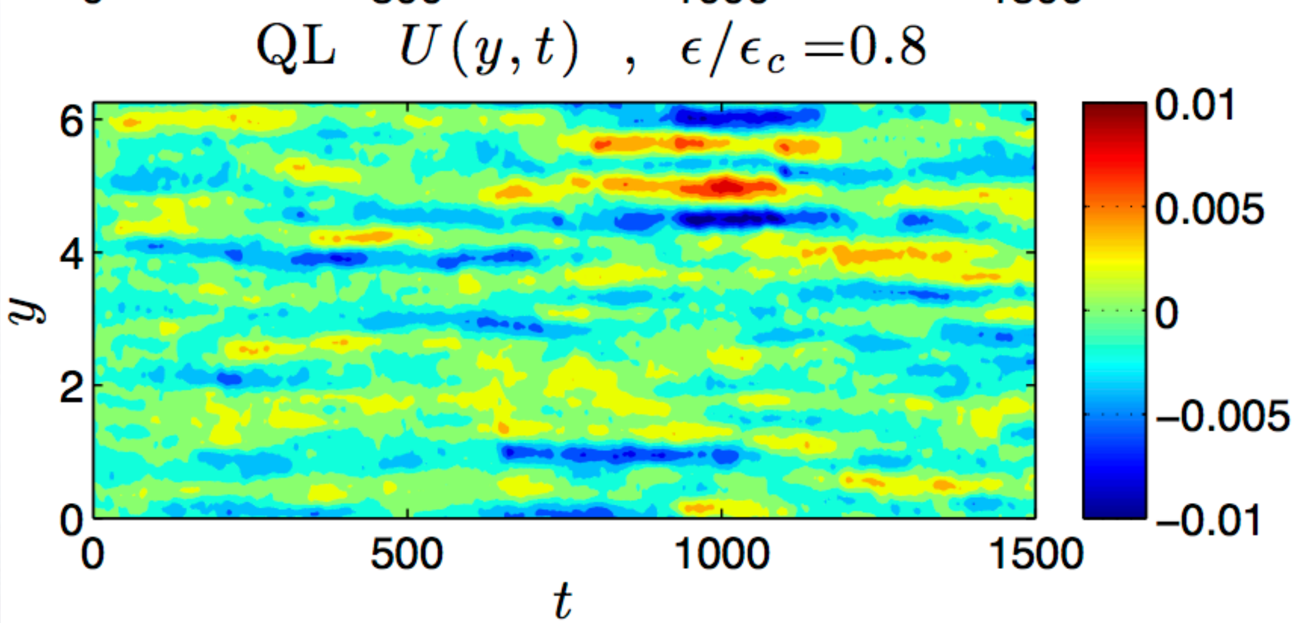
IRFh, $r = r_{\mathrm{m}} = 10^{-2}$



Stochastically excited stable SSST eigenmodes

(Latent jets in ocean: Berloff et. al. 2009, Berloff et. al. 2011, Cravatte et. al. 2012)



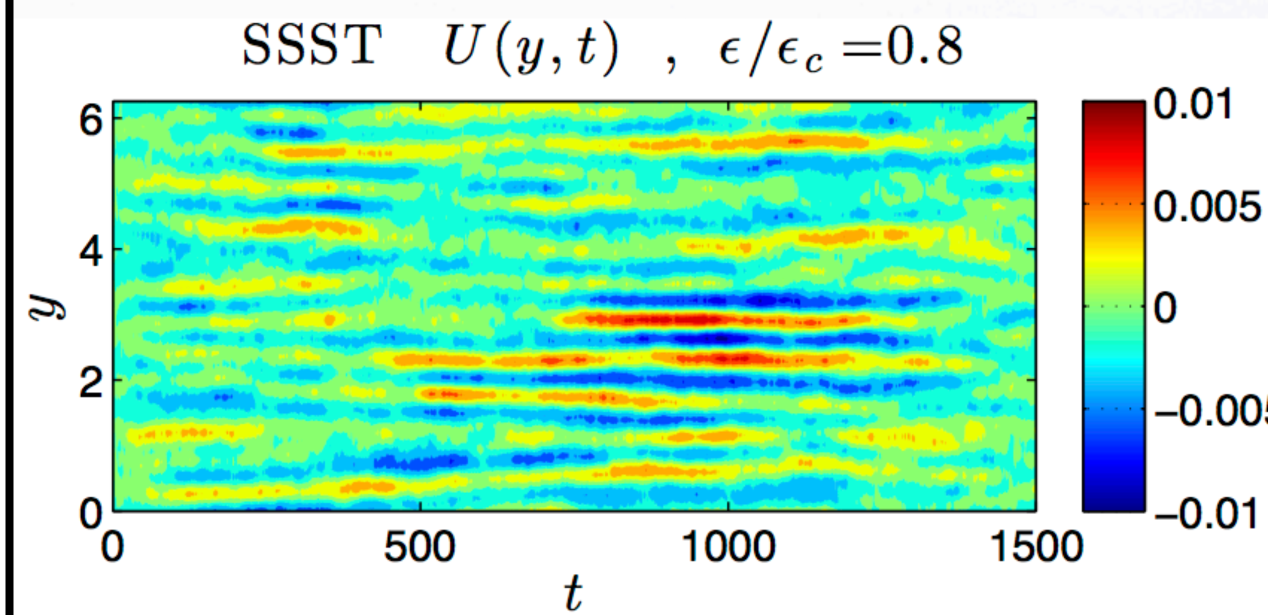


$$U(y,t) = \mathrm{Re}\left[\sum_{n=1}^{15} \alpha_n(t) e^{iny}\right]$$

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \sigma_n \, \alpha_n + \xi(t)$$

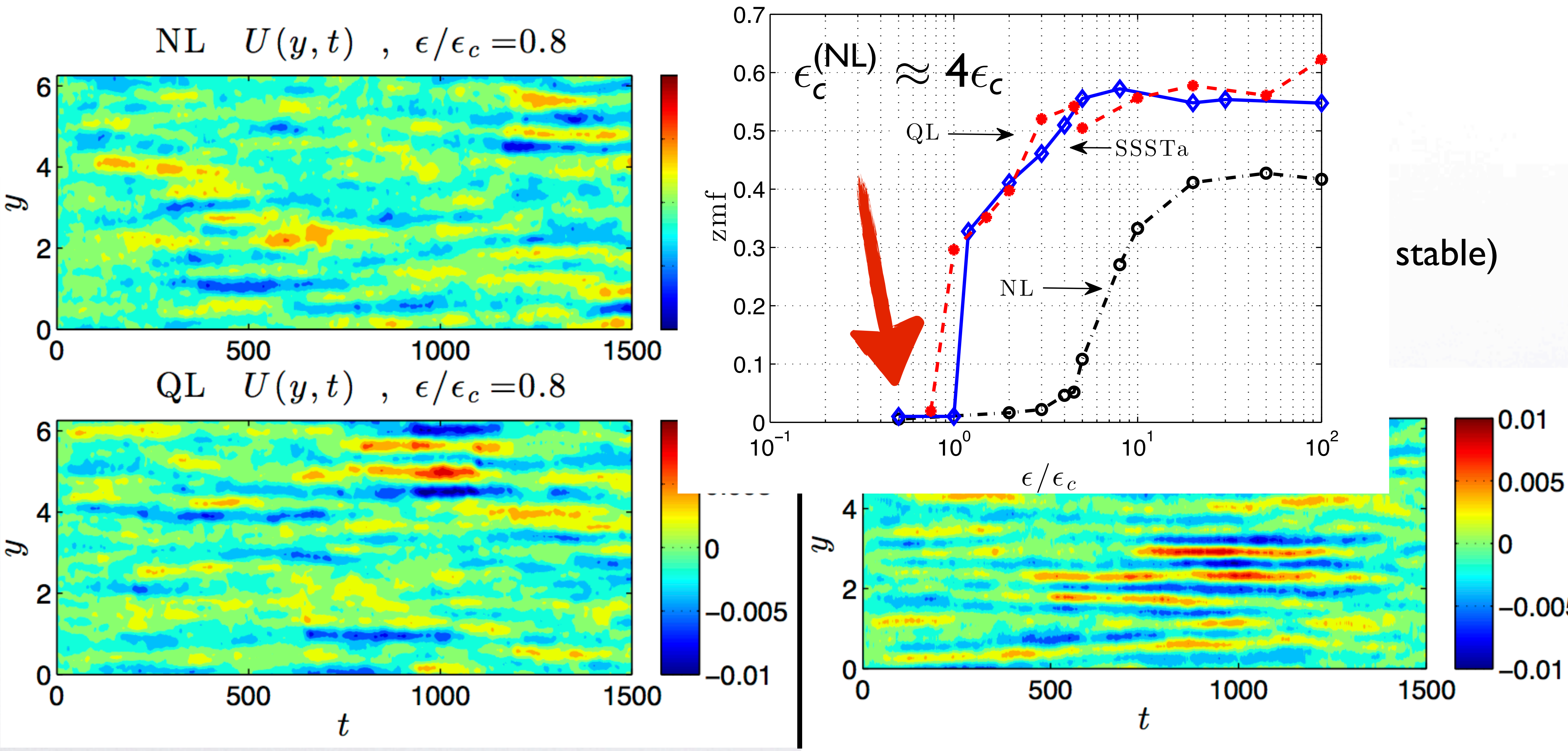
σ_n: SSST growth rates (least stable)

$\xi(t)$: delta-correlated noise



Stochastically excited stable SSST eigenmodes

(Latent jets in ocean: Berloff et. al. 2009, Berloff et. al. 2011, Cravatte et. al. 2012)



stable)

Stochastically excited stable SSST eigenmodes

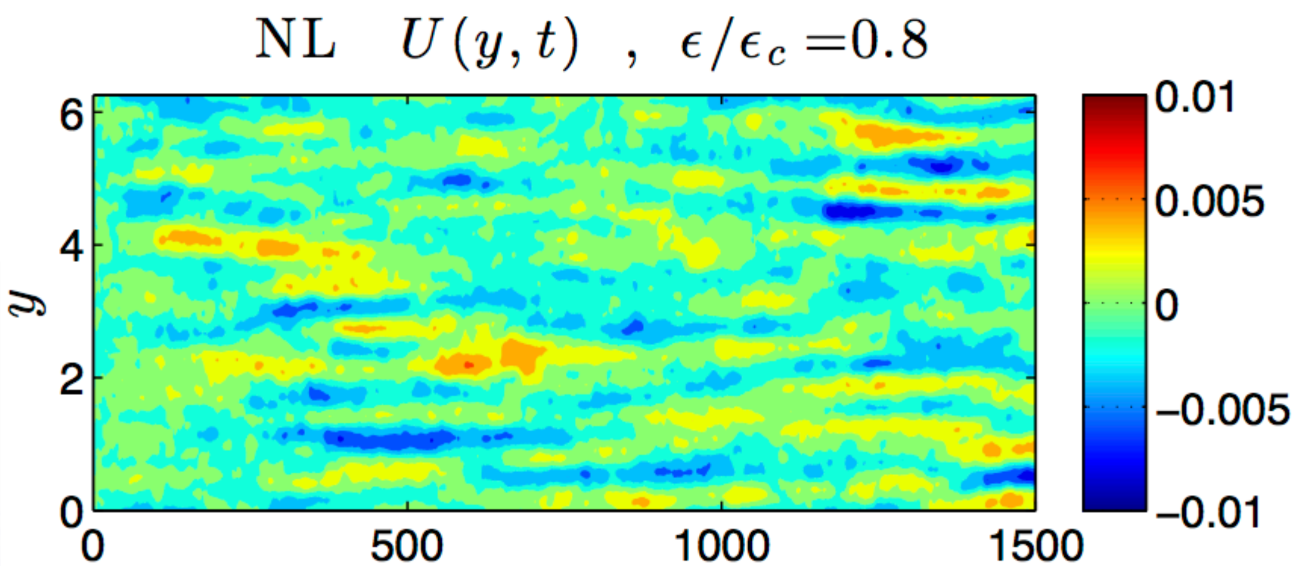
(Latent jets in ocean: Berloff et. al. 2009, Berloff et. al. 2011, Cravatte et. al. 2012)

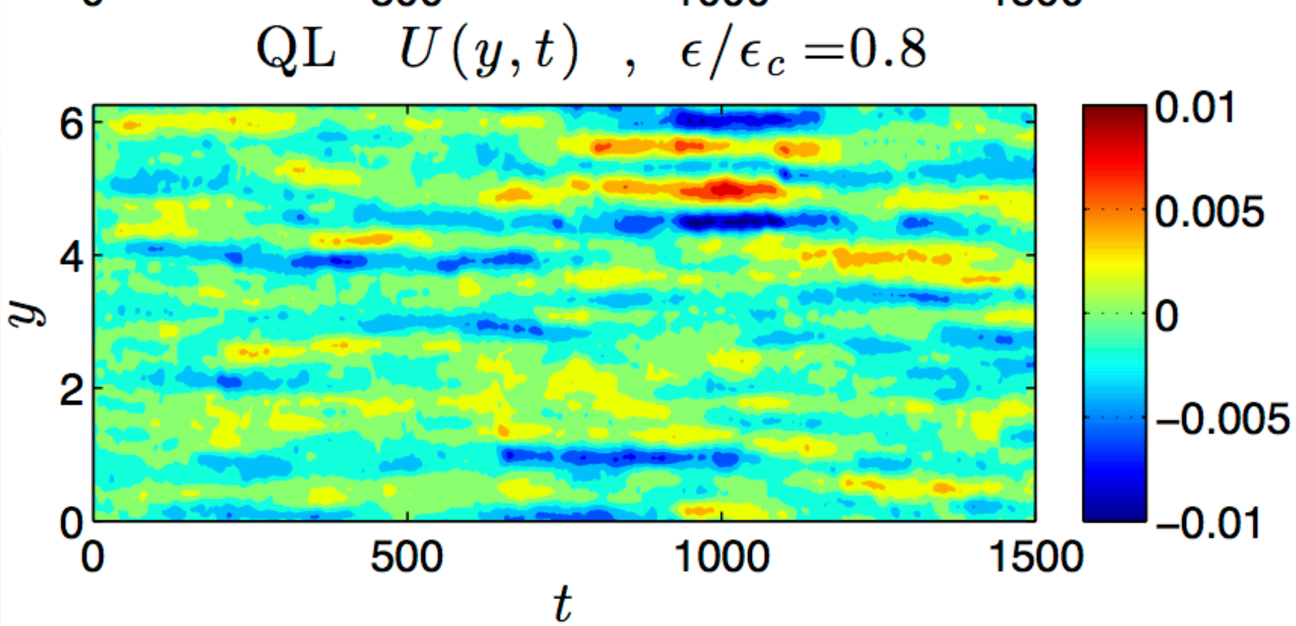
$$U(y,t) = \mathrm{Re}\left[\sum_{n=1}^{15} \alpha_n(t) e^{iny}\right]$$

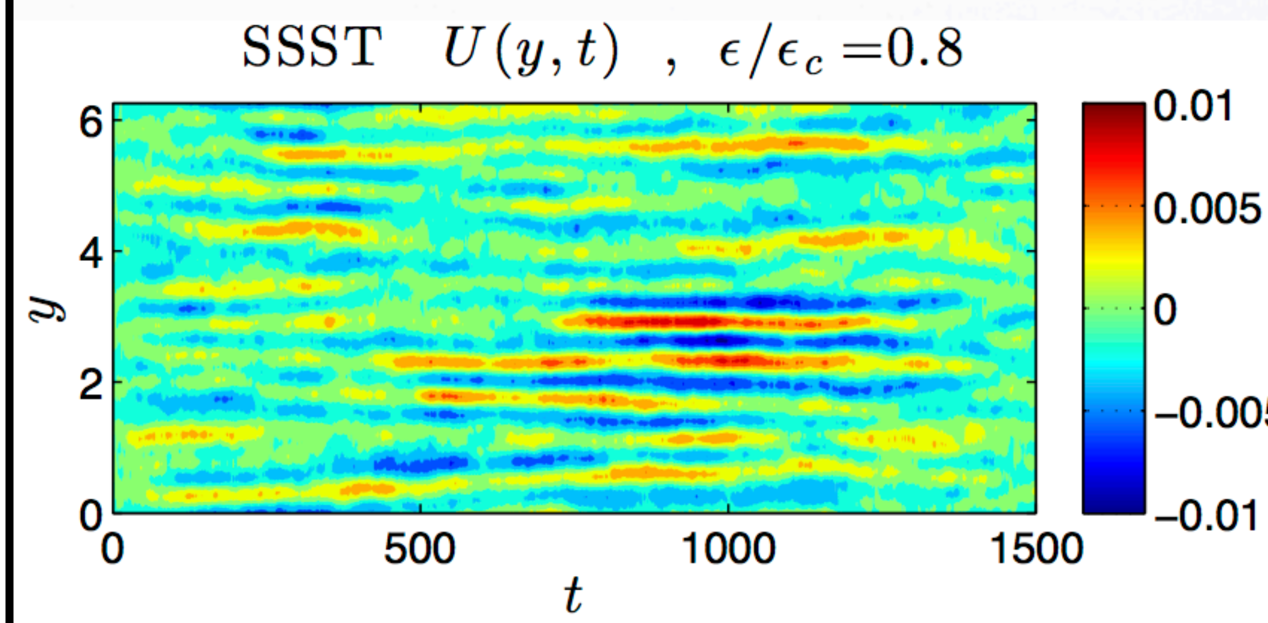
$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \sigma_n \, \alpha_n + \xi(t)$$

σ_n: SSST growth rates (least stable)

$\xi(t)$: delta-correlated noise



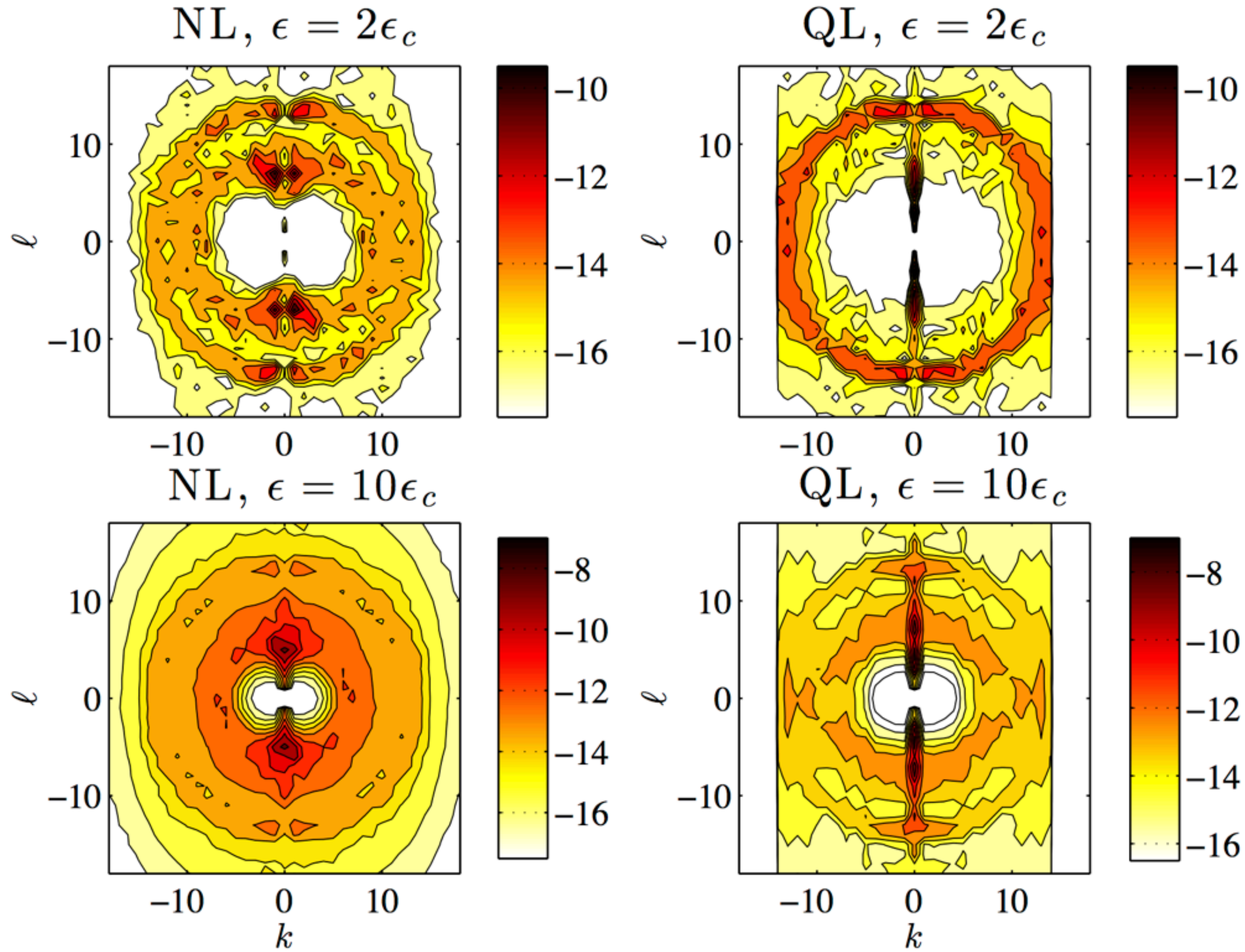




Zonons at NL

energy spectrum

IRFh, $r = r_\mathrm{m} = 10^{-2}$

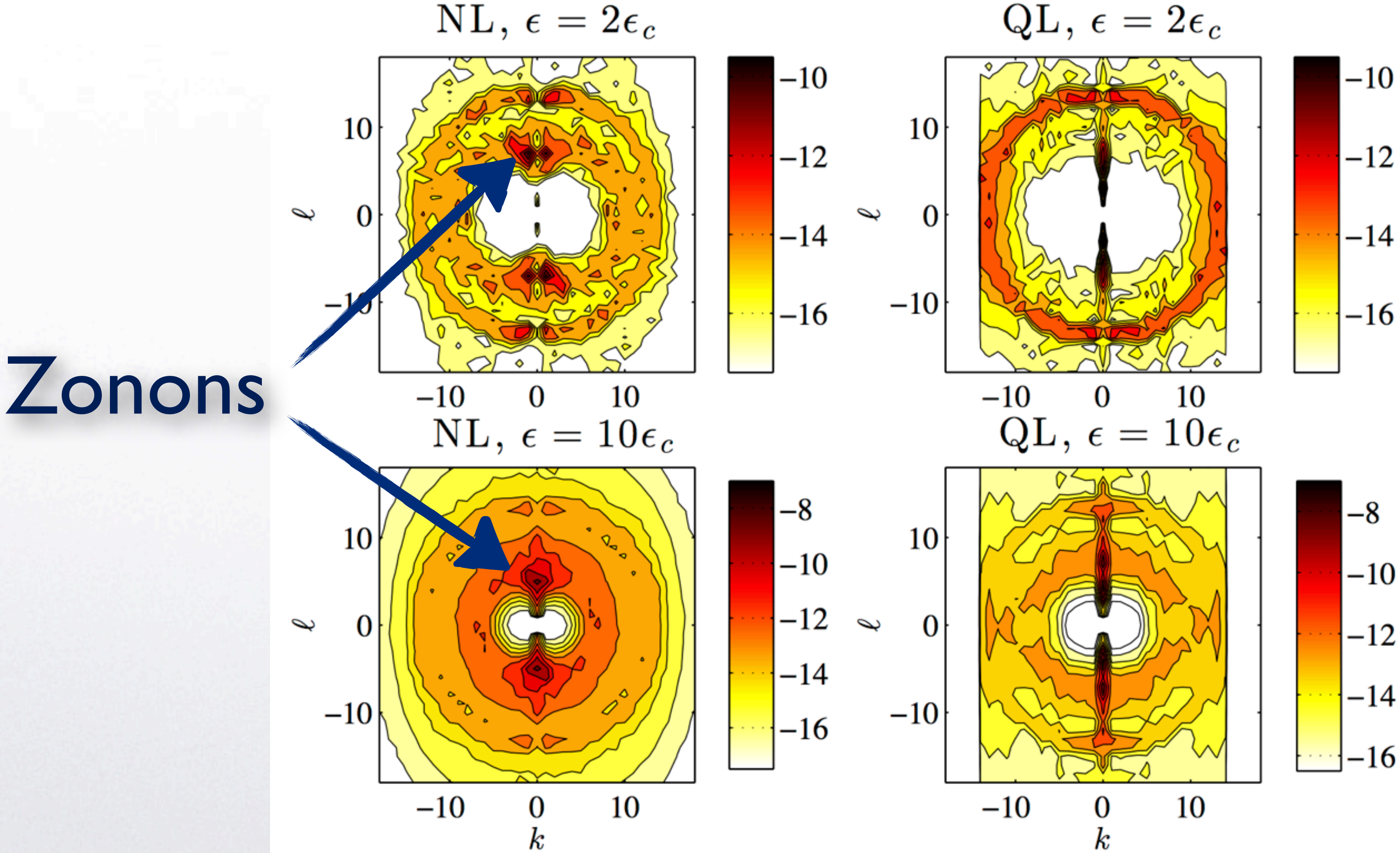


QL / SSST do not allow the the interaction of the turbulence with coherent non-zonal structures

Zonons at NL

energy spectrum

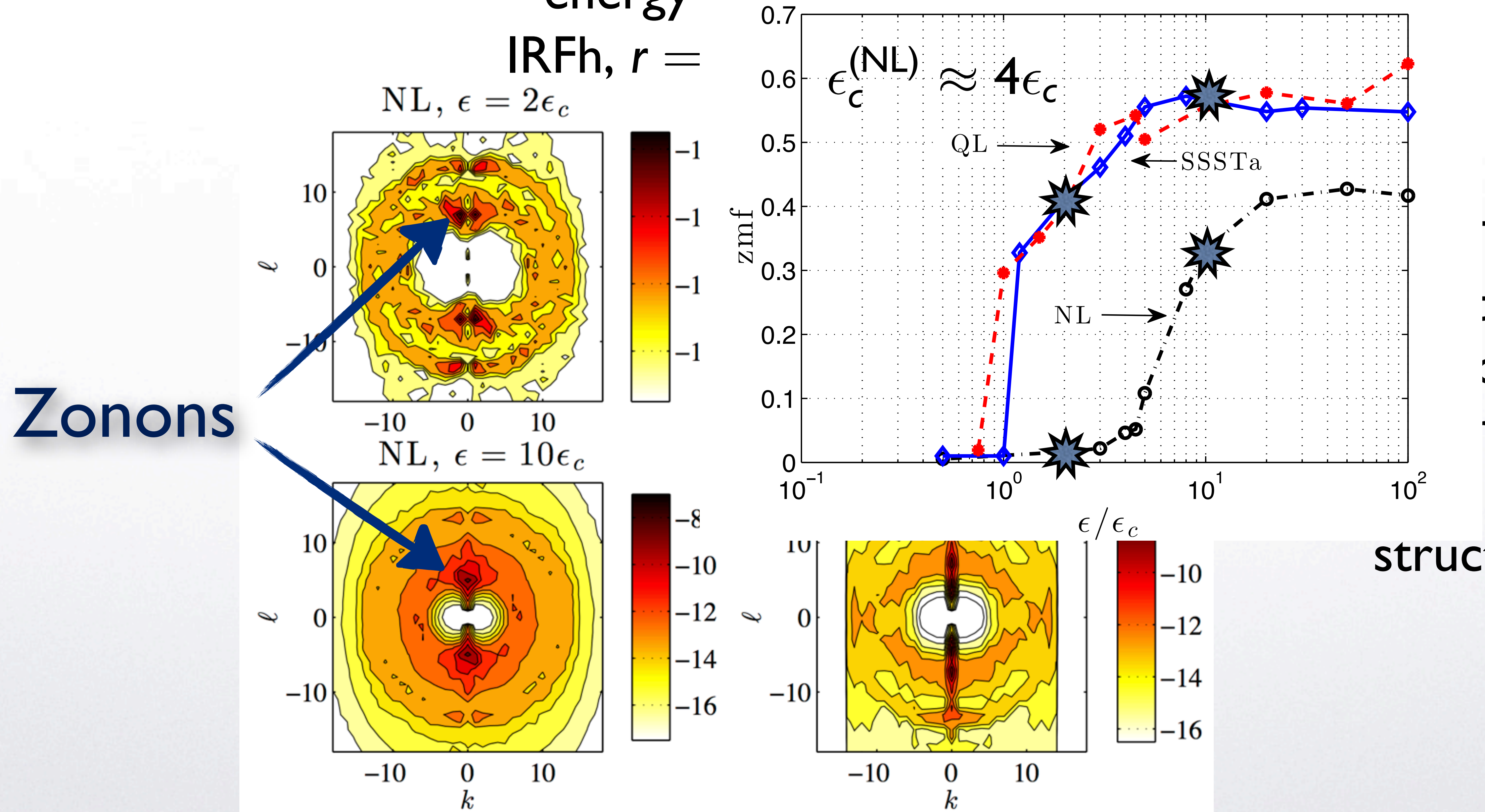
IRFh, $r = r_m = 10^{-2}$



QL / SSST do not allow the the interaction of the turbulence with coherent non-zonal structures

Zonons at NL

energy
IRFh, $r =$

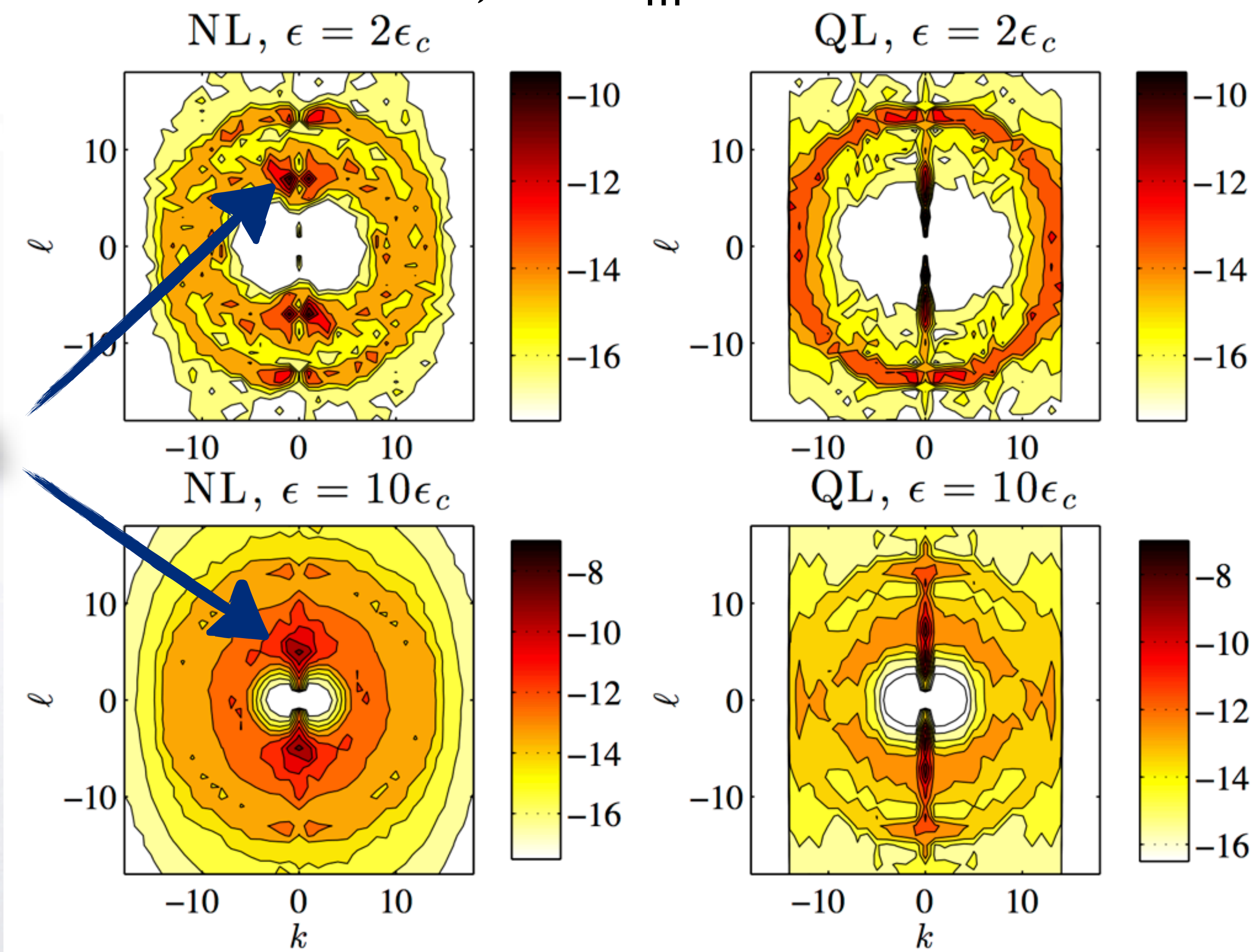


T do not
the the
on of the
nce with
non-zonal
structures

Zonons at NL

energy spectrum

IRFh, $r = r_{\mathrm{m}} = 10^{-2}$



Zonons

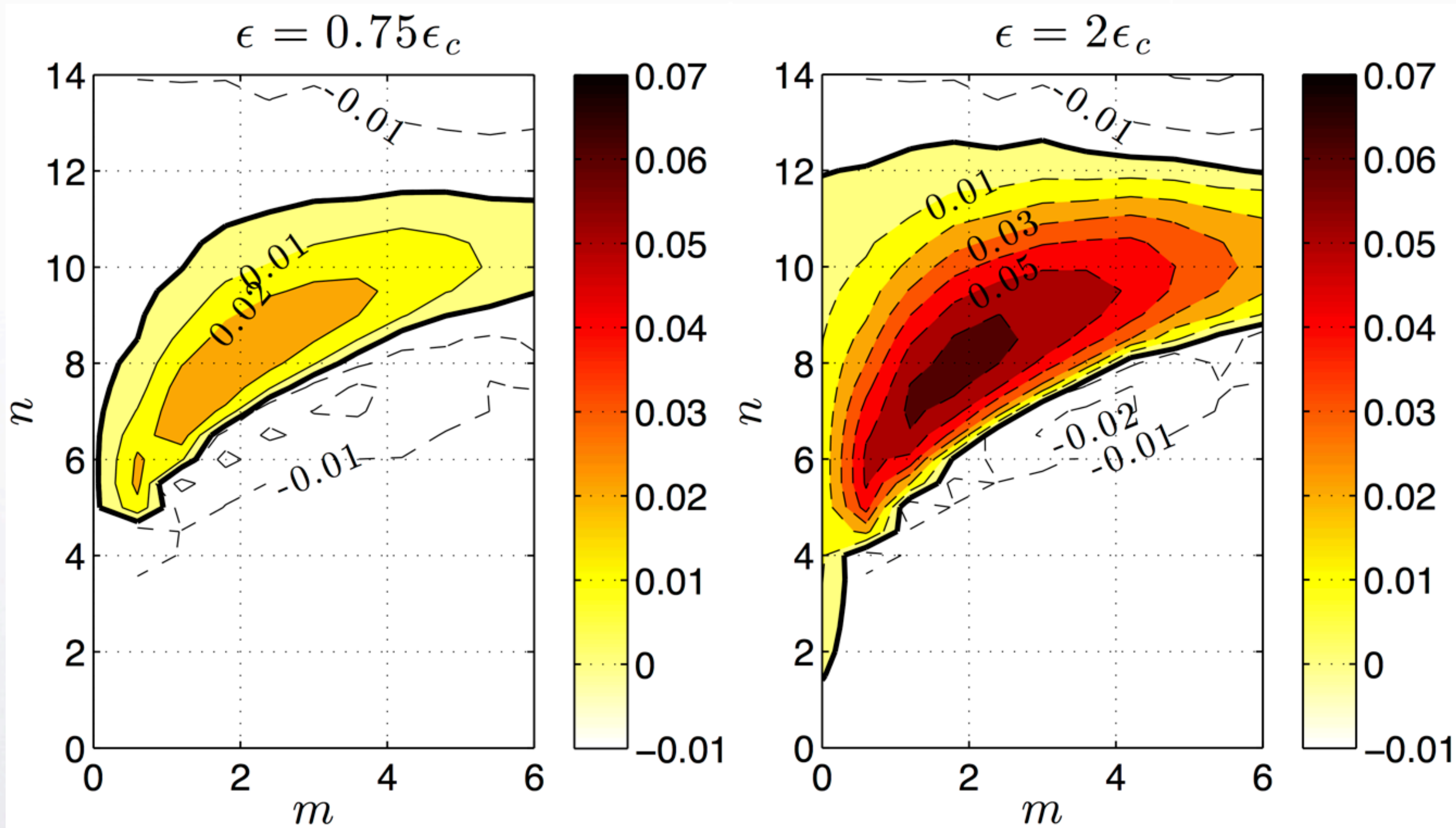
QL / SSST do not allow the the interaction of the turbulence with coherent non-zonal structures

Non-zonal instabilities to homogeneous equilibrium

perturbations of the homogeneous equilibrium in form $\delta U = \mathrm{e}^{\mathrm{i}(mx+ny)}$

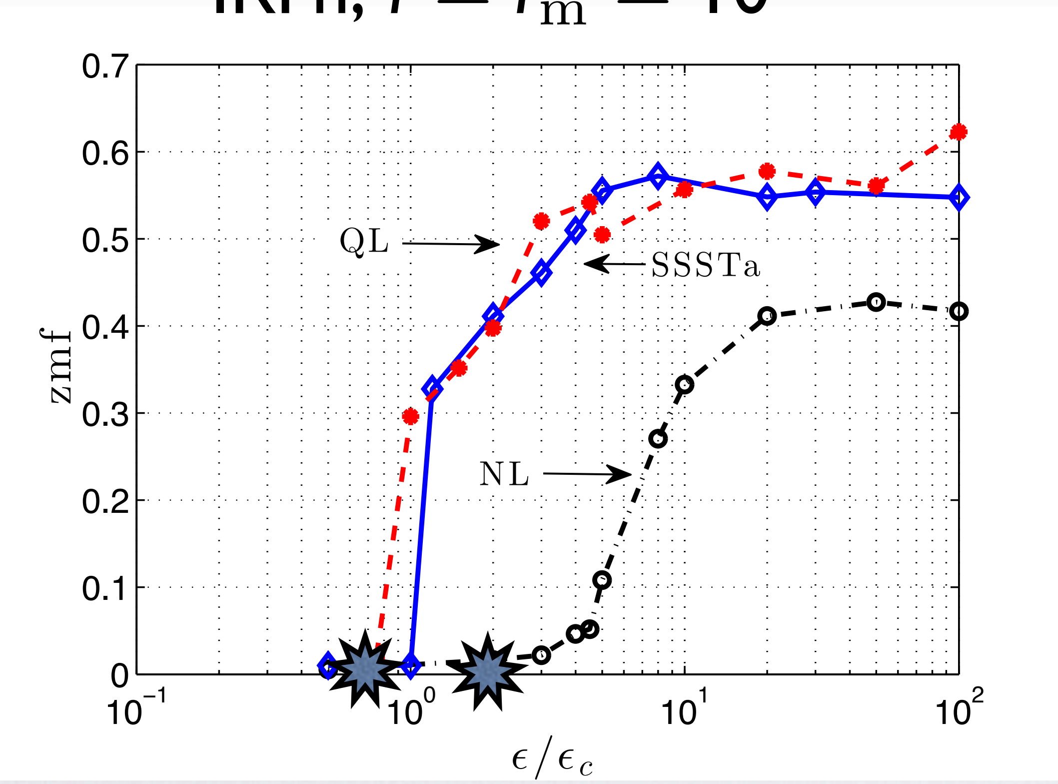
for these parameters non-zonal perturbations are more SSST unstable than jets ($m = 0$)

IRFh, $r = r_\mathrm{m} = 10^{-2}$

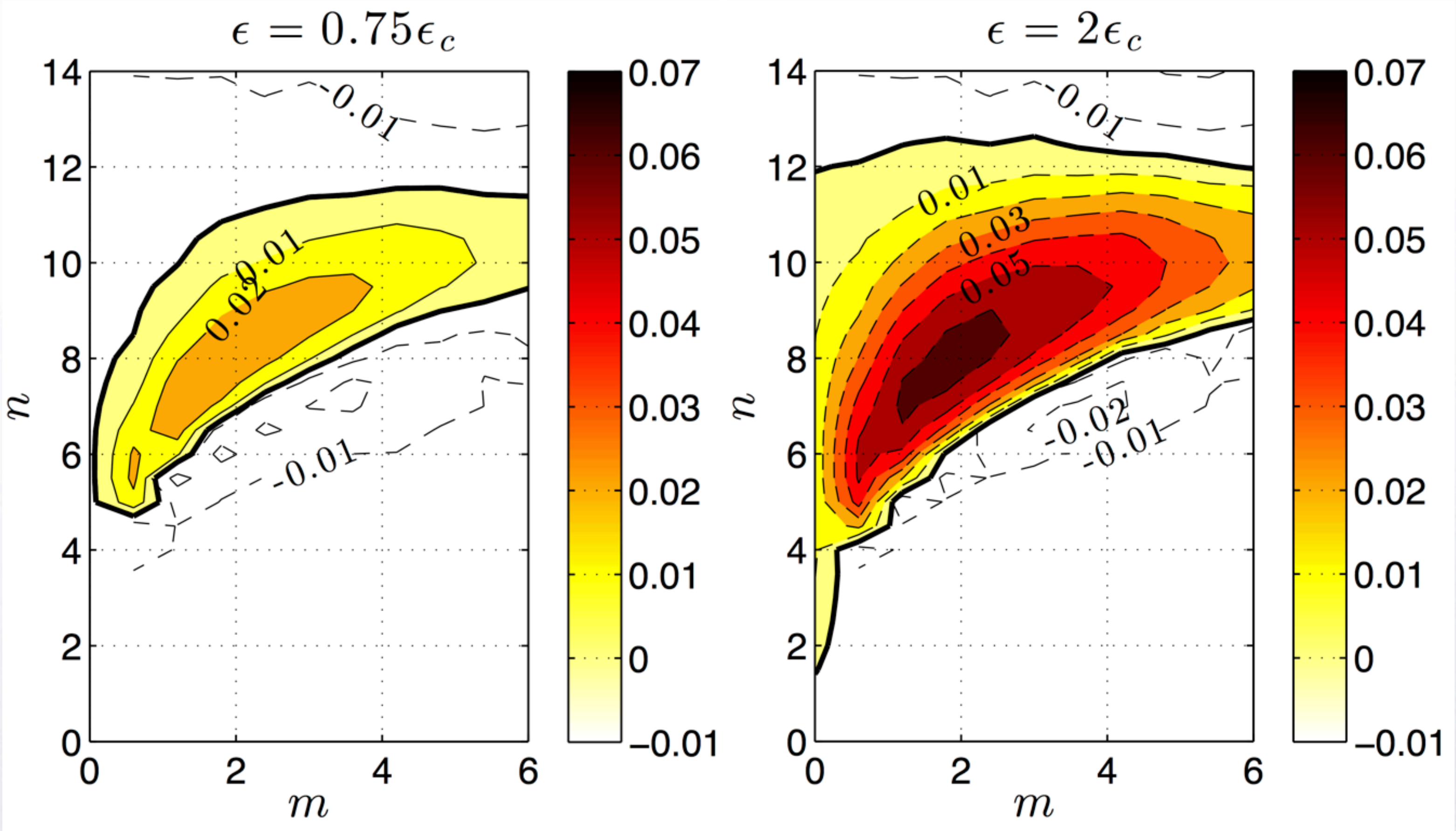


Non-zonal instabilities to homogeneous equilibrium

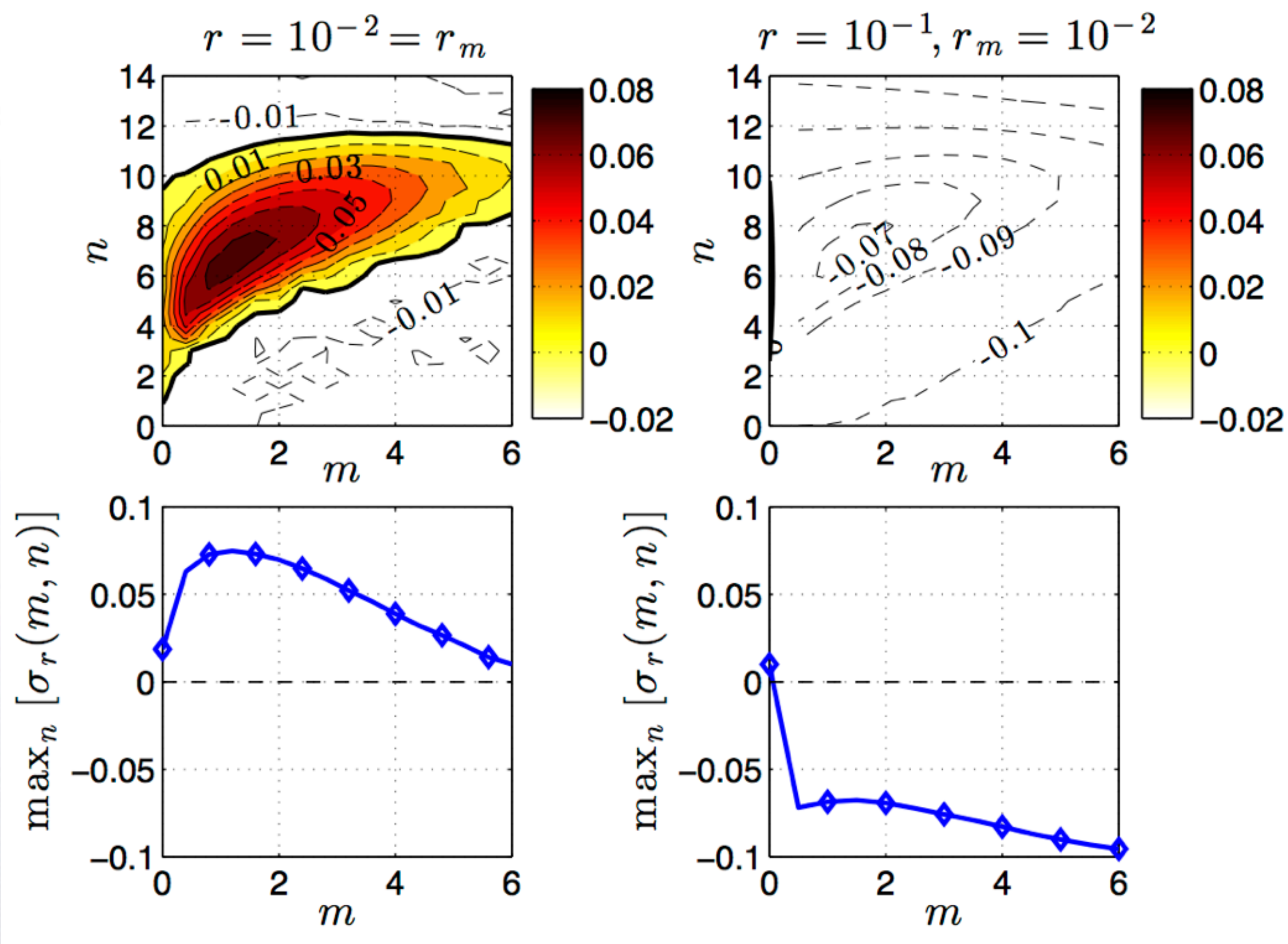
IRFh, $r = r_{\mathrm{m}} = 10^{-2}$



IRFh, $r = r_{\mathrm{m}} = 10^{-2}$

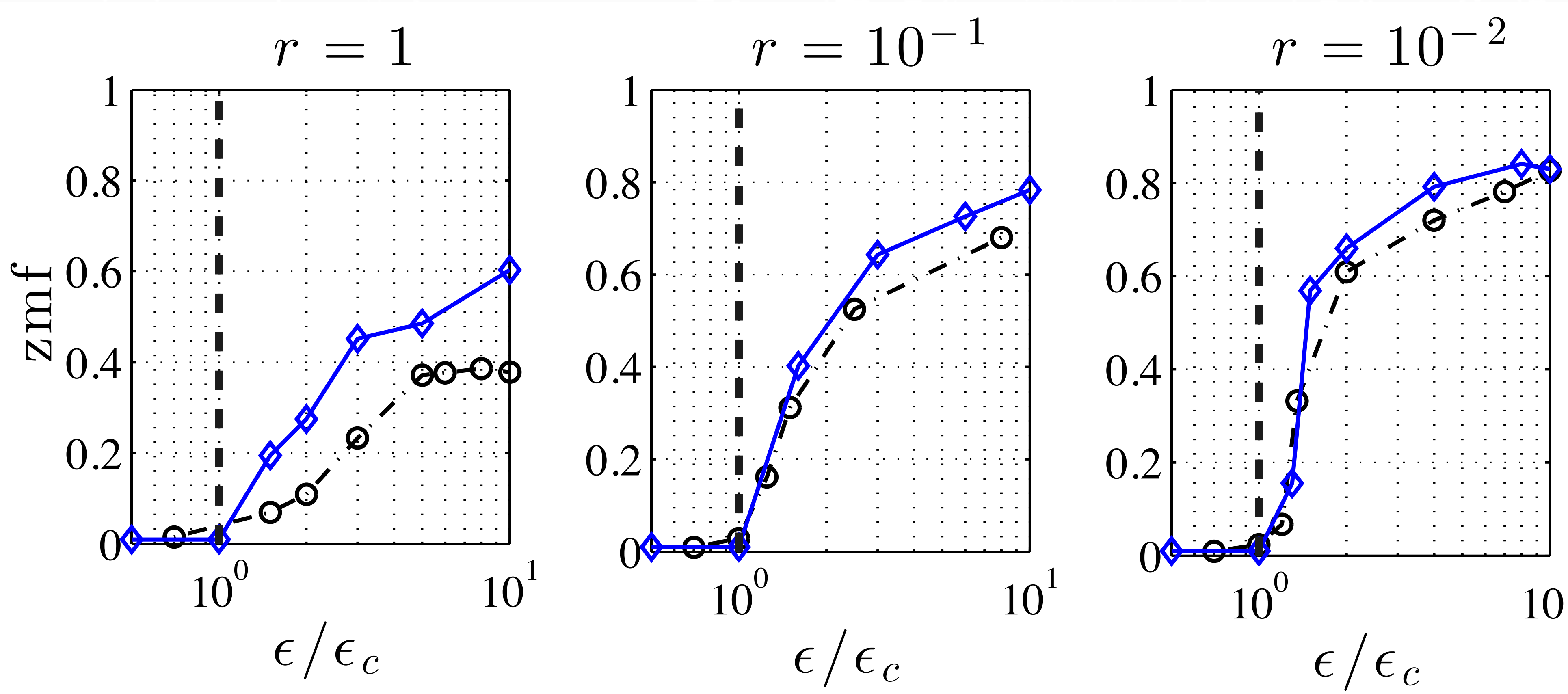


Depression of non-zonal instabilities using $r_m = r/10$



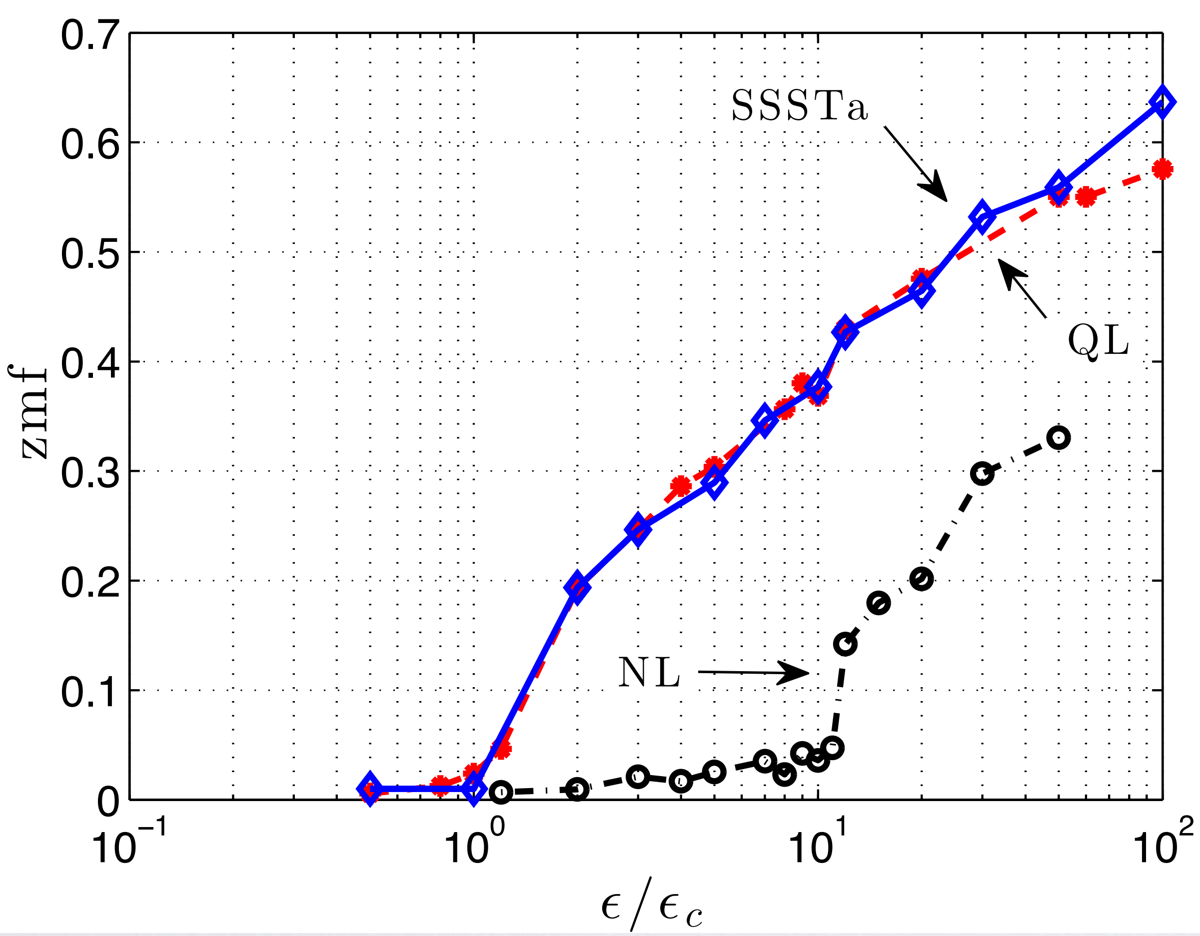
SSST captures the NL bifurcation

NIF, $r_\mathrm{m} = r/10$

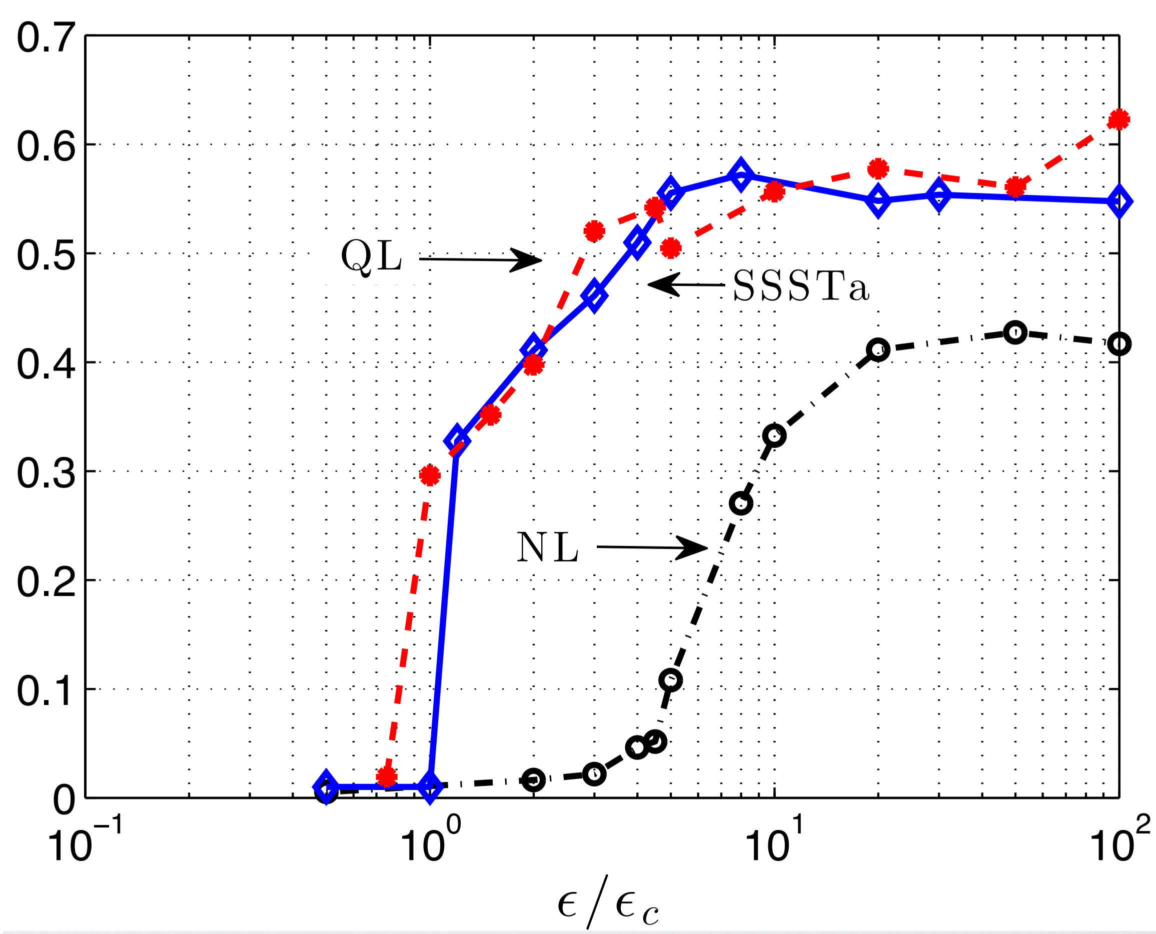


Bifurcation diagrams for $r = r_{\mathrm{m}}$ (revisited)

NIF, $r = r_{\mathrm{m}} = 10^{-2}$



IRFh, $r = r_{\mathrm{m}} = 10^{-2}$



Bifurcation diagrams for $r = r_\mathrm{m}$ (revisited)

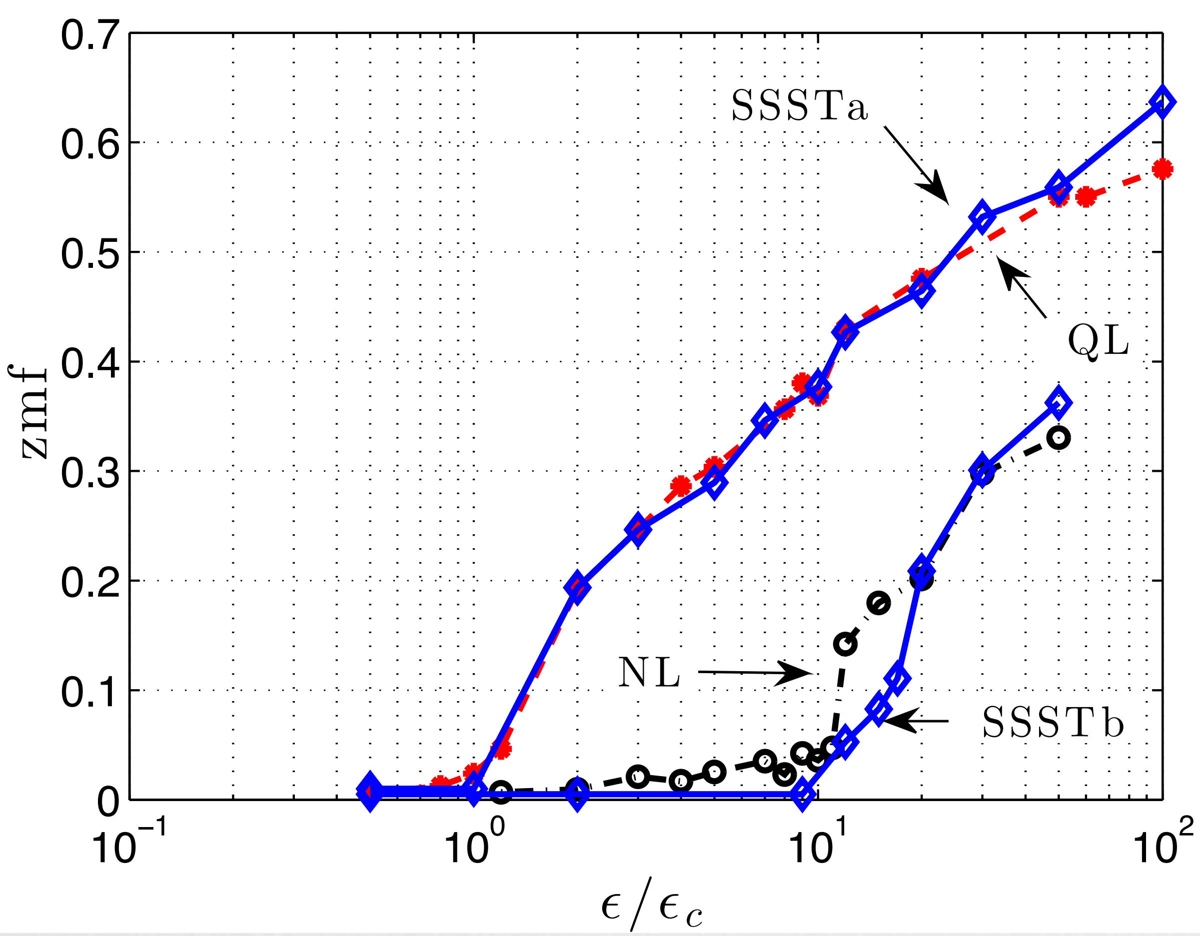
If we use the non-coherent part of the spectrum of the NL statistical equilibrium to prescribe the stochastic forcing covariance, then SSST predicts the correct value for jet emergence

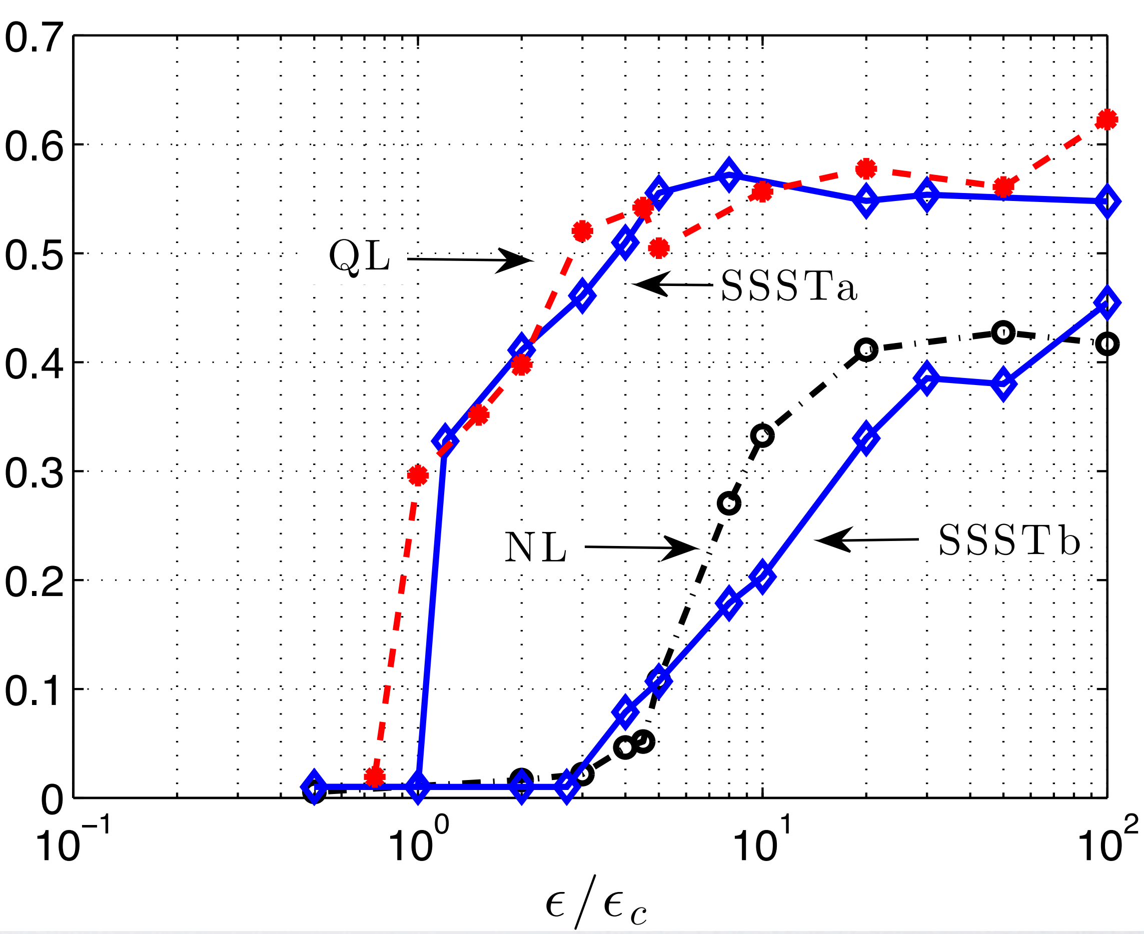
Moreover, the SSST equilibrium zonal flow amplitude agree with the ones from NL integrations

Bifurcation diagrams for $r = r_{\mathrm{m}}$ (revisited)

NIF, $r = r_{\mathrm{m}} = 10^{-2}$

IRFh, $r = r_{\mathrm{m}} = 10^{-2}$



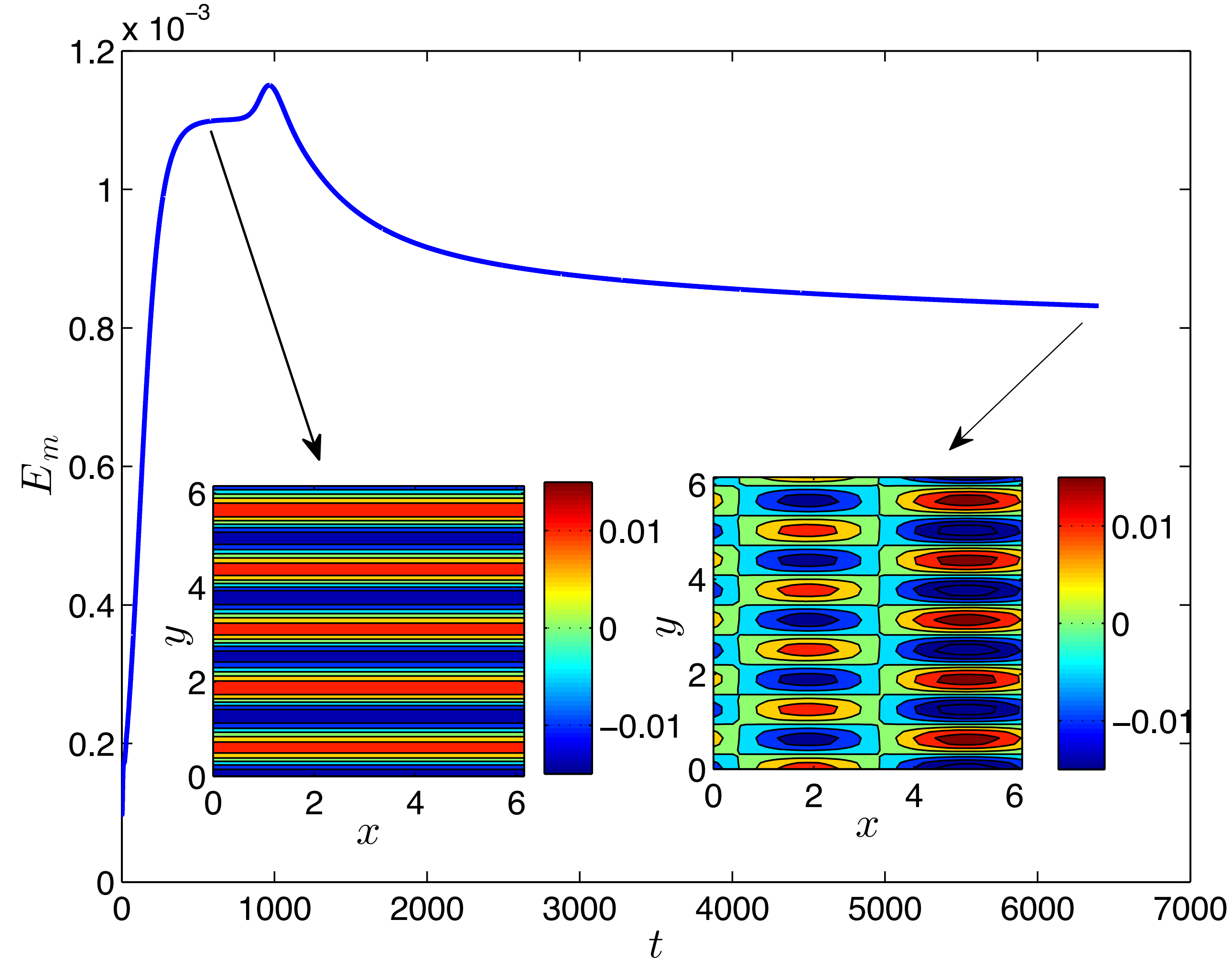


Generalized SSST admits non-zonal equilibria

IRFh at $\epsilon = 2\epsilon_c$

SSST at $\epsilon = 2\epsilon_c$ forms jets

generalized SSST initially goes goes to a jet equilibrium, which is SSST unstable and switches to a NZCS

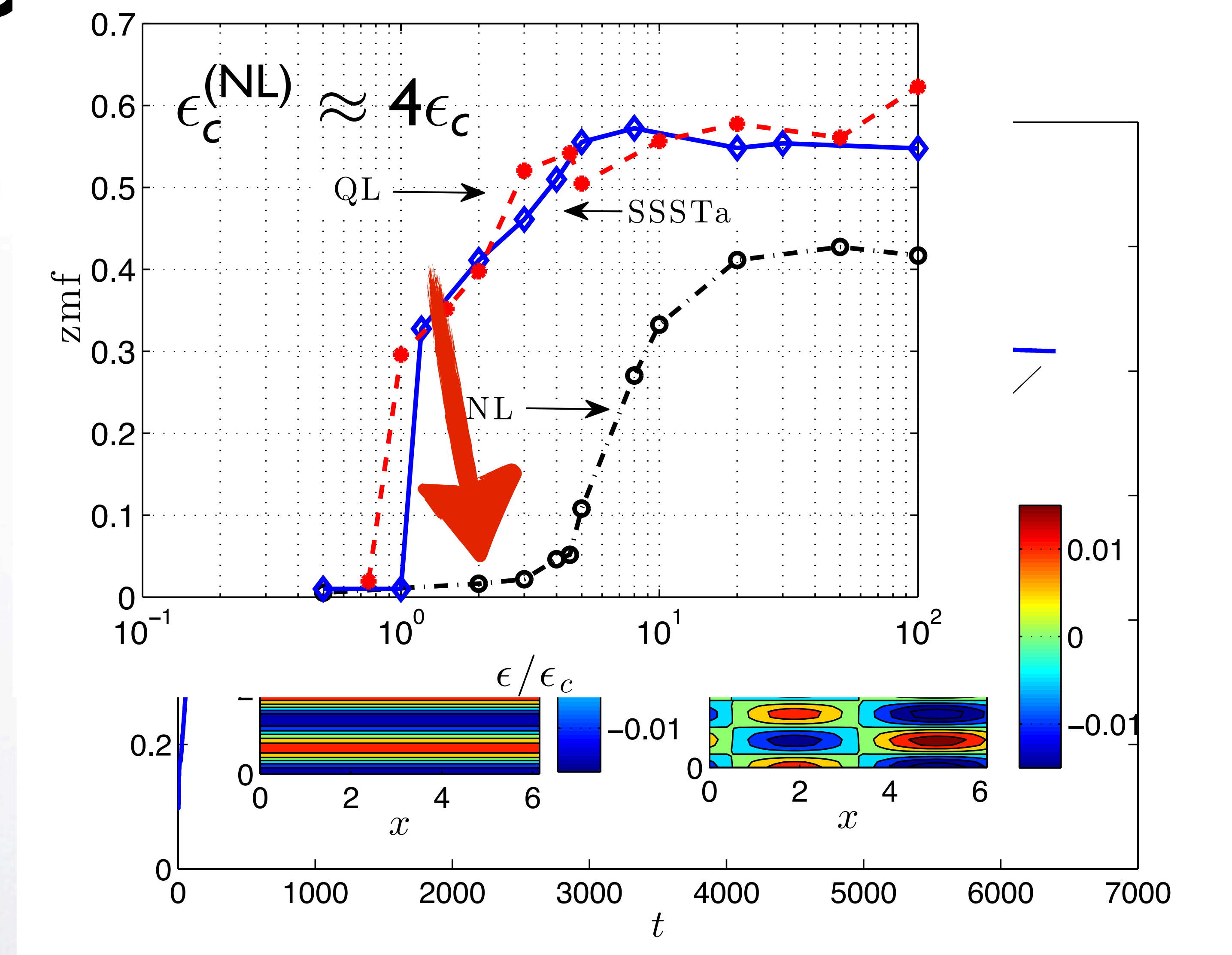


Generalized SSST admits non-zonal equilibria

IRFh at $\epsilon = 2\epsilon_c$

SSST at $\epsilon = 2\epsilon_c$ forms jets

generalized SSST initially goes goes to a jet equilibrium, which is SSST unstable and switches to a NZCS

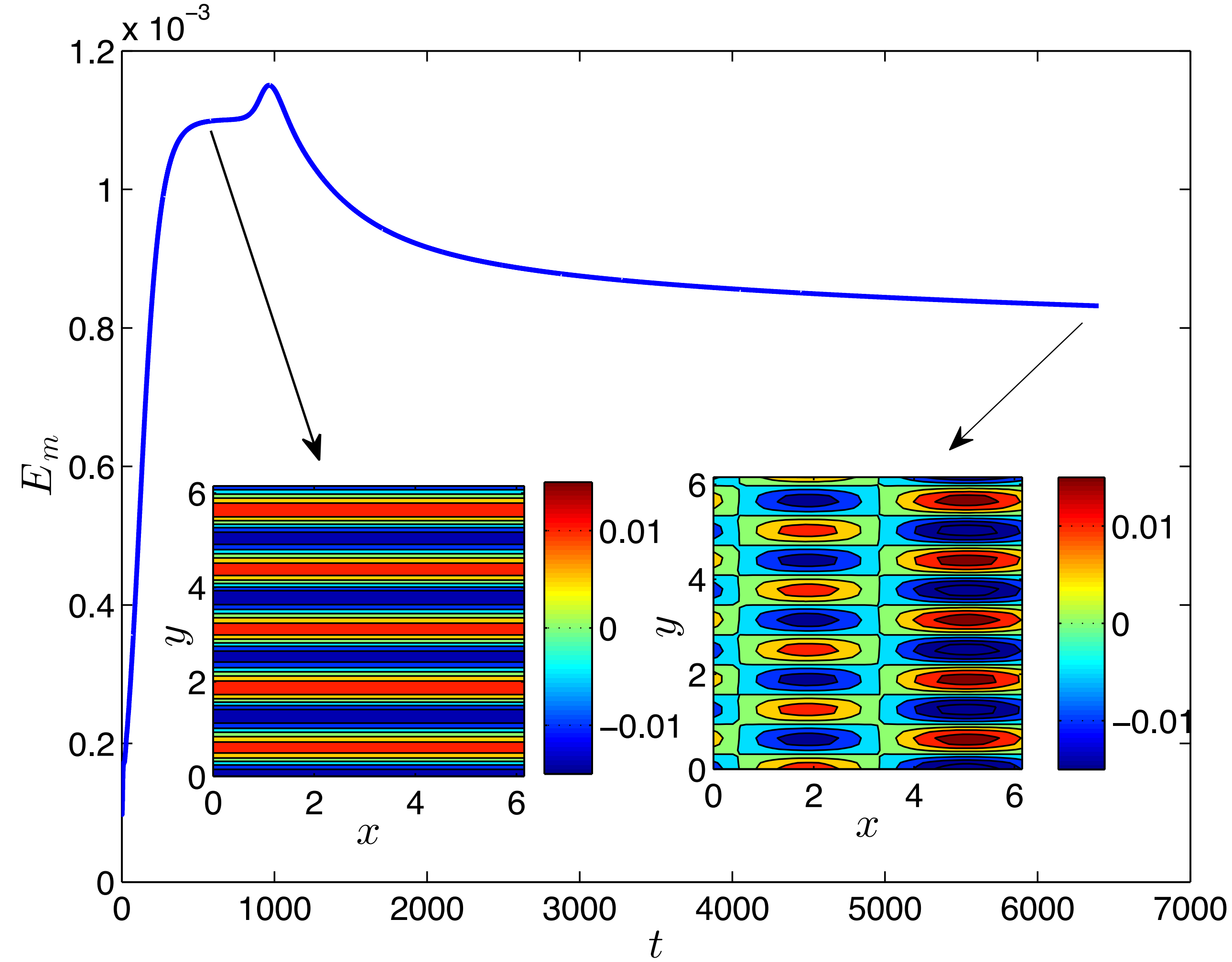


Generalized SSST admits non-zonal equilibria

IRFh at $\epsilon = 2\epsilon_c$

SSST at $\epsilon = 2\epsilon_c$ forms jets

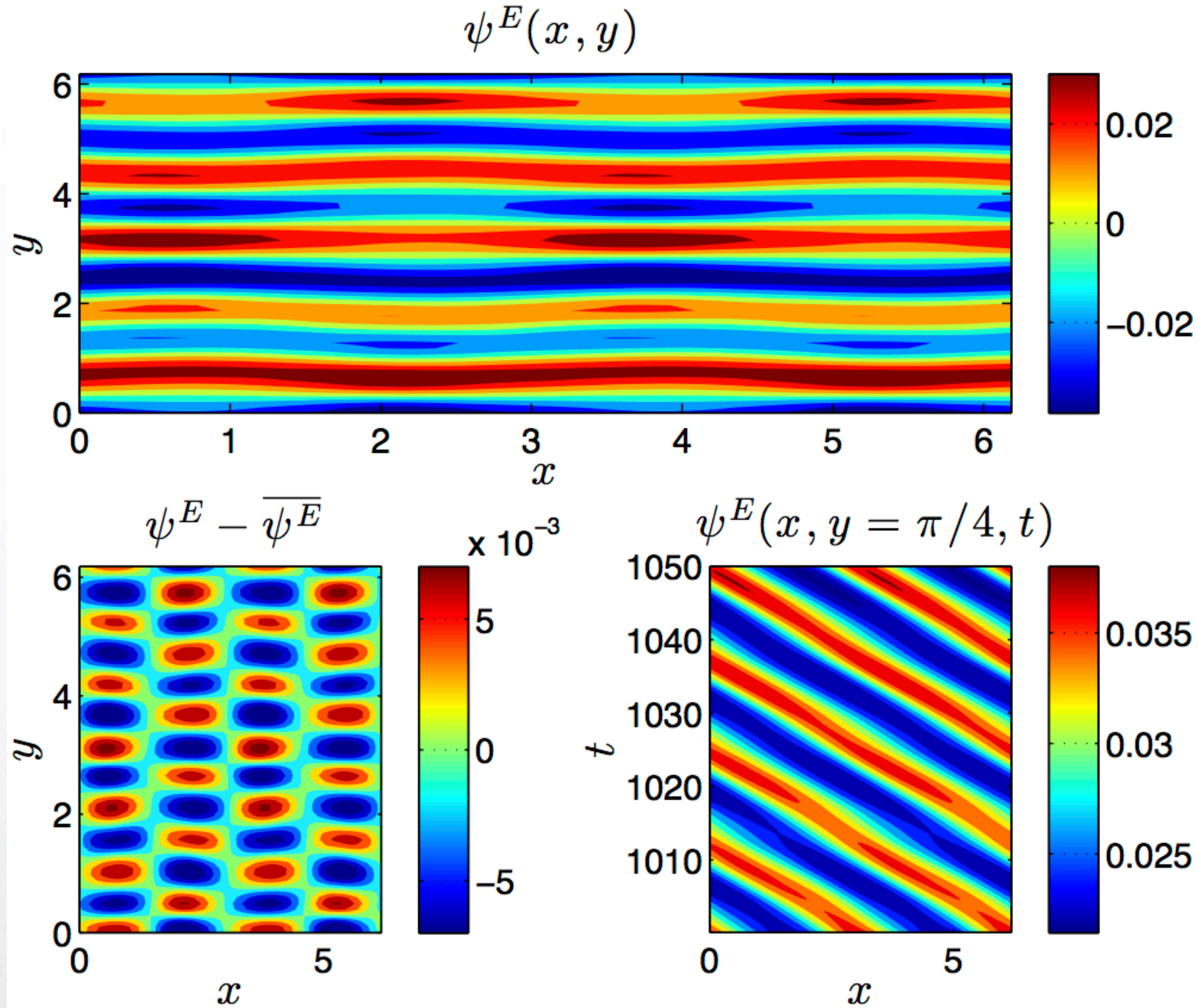
generalized SSST initially goes goes to a jet equilibrium, which is SSST unstable and switches to a NZCS



Generalized SSST equilibria show both zonal and non-zonal spectral peaks

IRFh at $\epsilon = 6\epsilon_c$

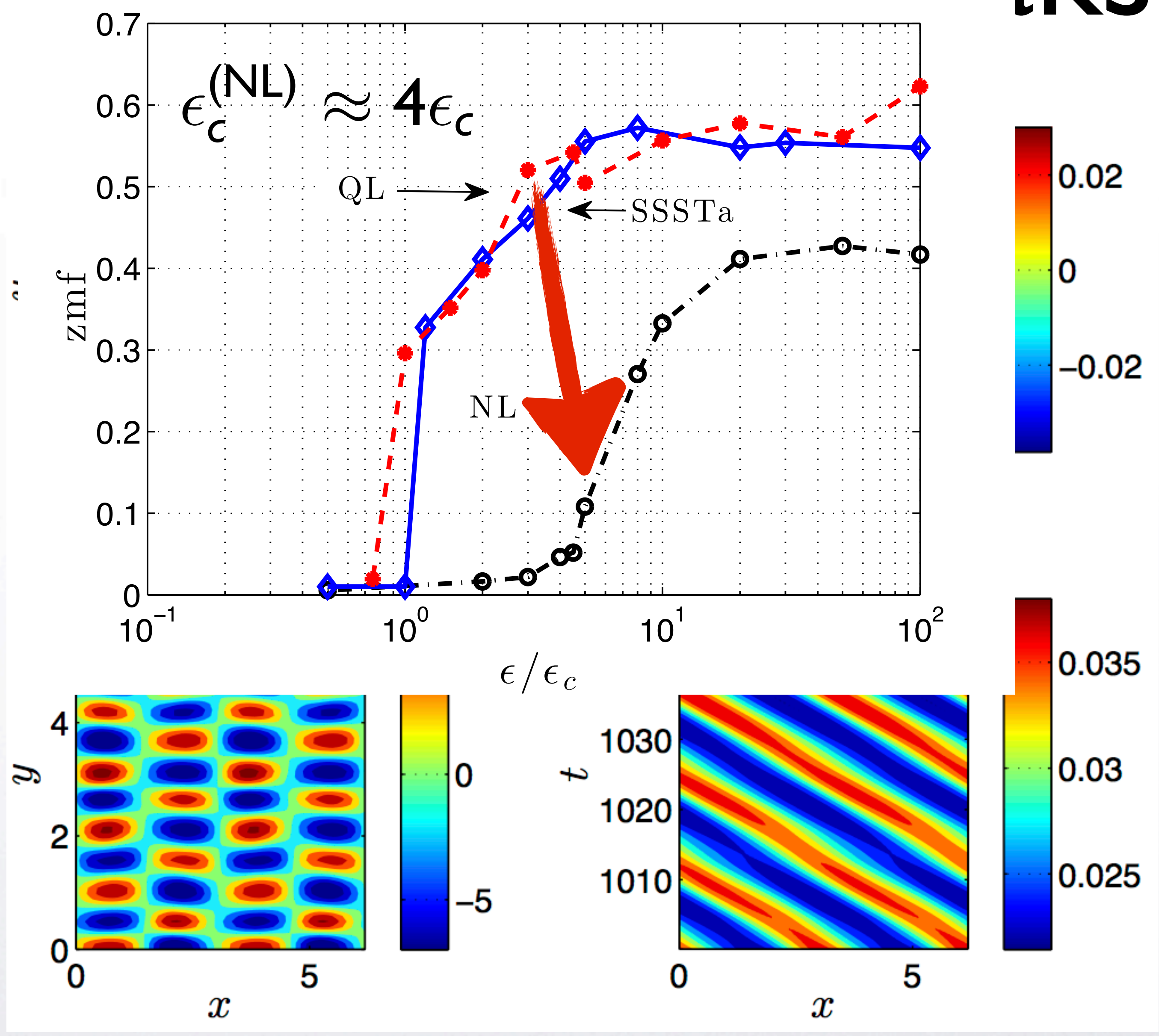
generalized SSST admits equilibria with zonal as well as non-zonal spectral components



Generalized SSST equilibria show both zonal and non-zonal spectral peaks

IRFh at $\epsilon = 6\epsilon_c$

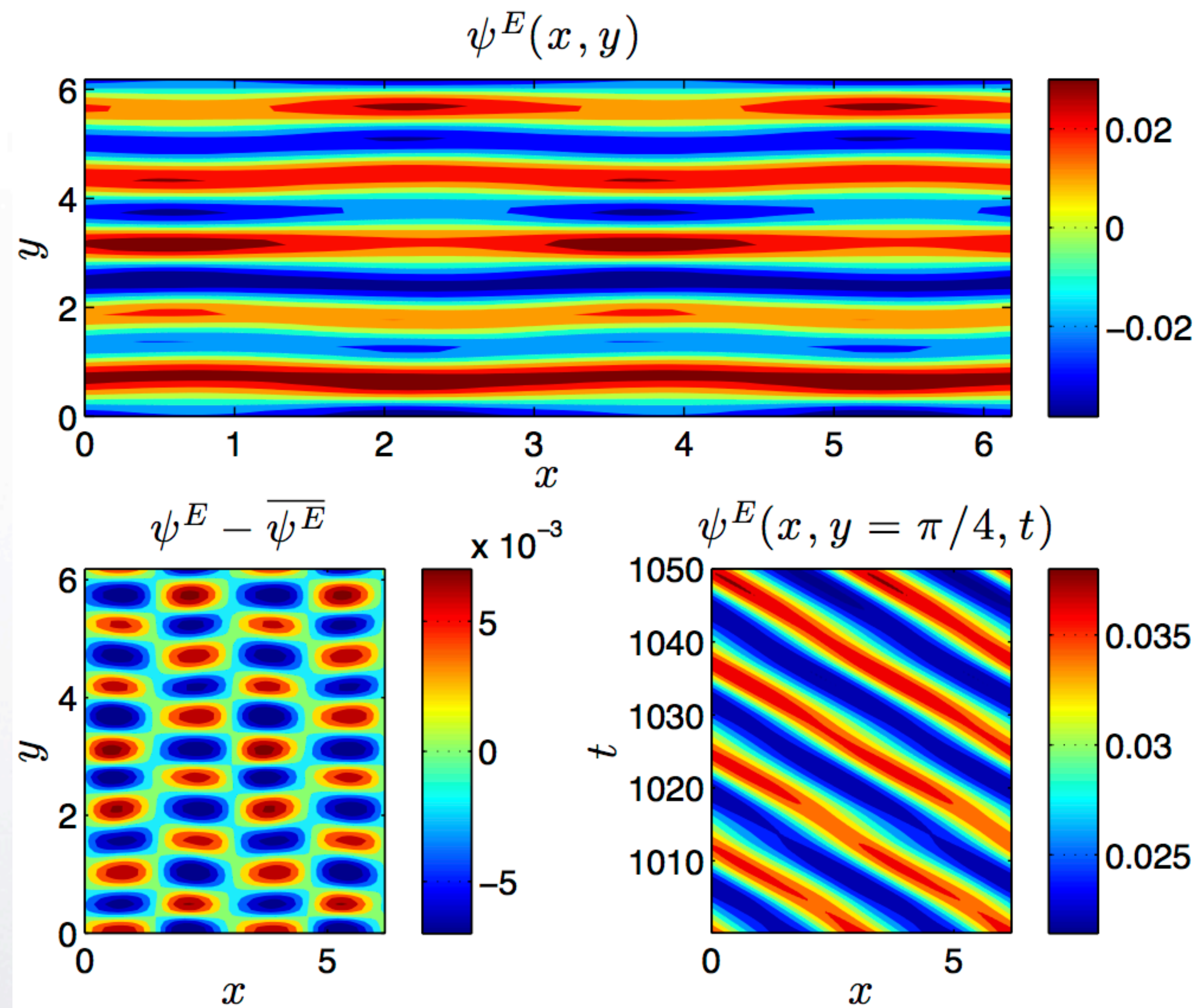
generalized SSST admits equilibria with zonal as well as non-zonal spectral components



Generalized SSST equilibria show both zonal and non-zonal spectral peaks

IRFh at $\epsilon = 6\epsilon_c$

generalized SSST admits equilibria with zonal as well as non-zonal spectral components



Things to discuss

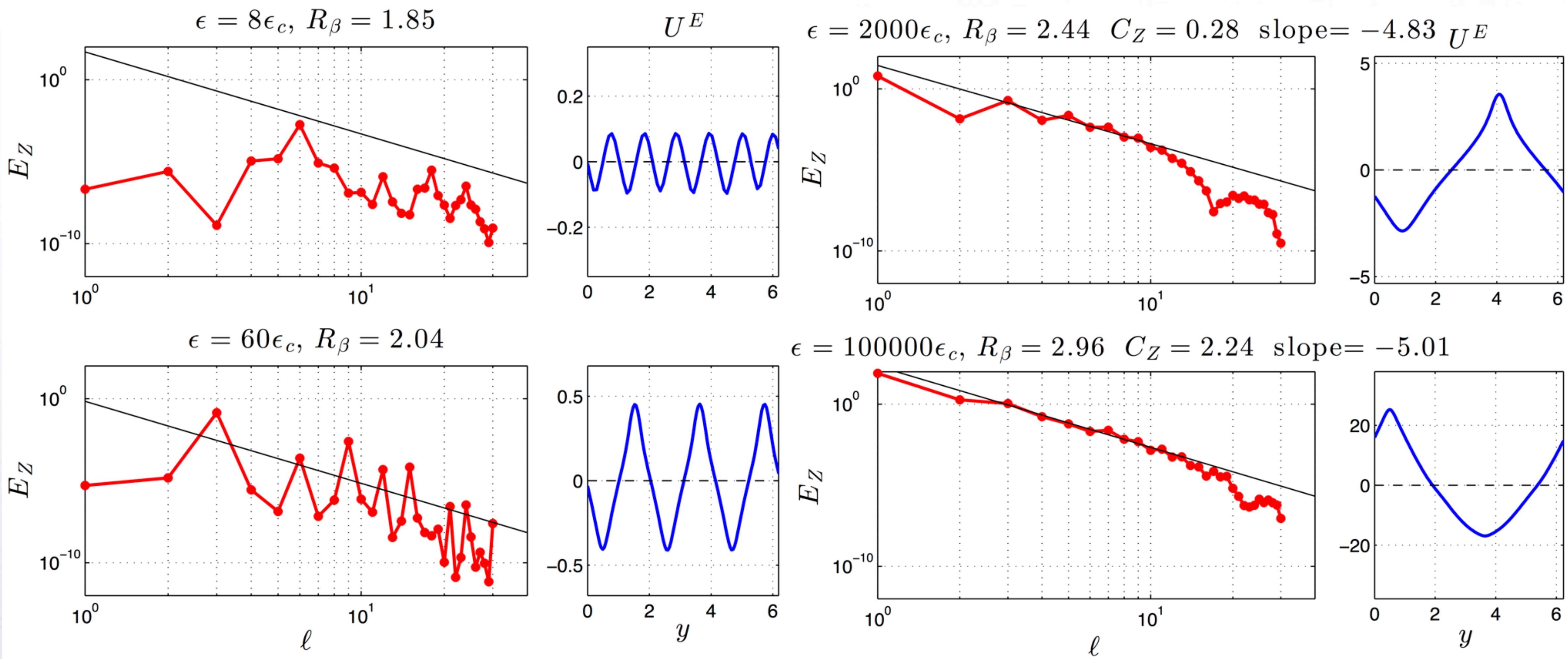
- What is the shape of the SSST mean flow equilibria? Does it capture the NL jet shape?

- Jet mergings / why do they occur?
  (is it because of U becomes hydrodynamically unstable?)

- PV staircases (?)

SSST equilibrium zonal flow spectral structure

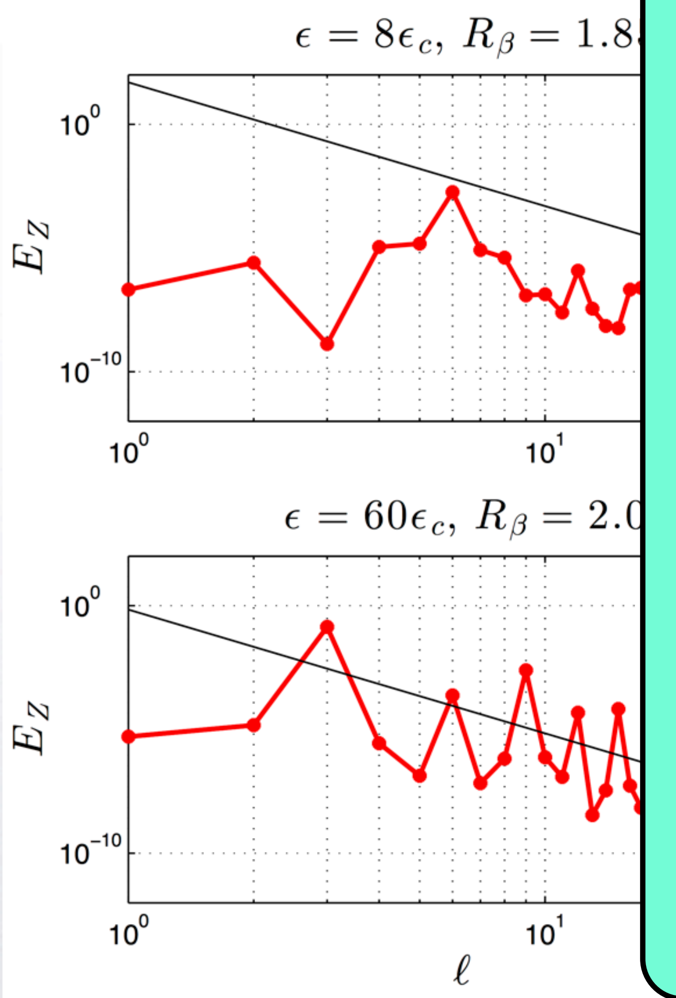
$$E_Z = E(k=0,\ell) = C_Z \beta^2 \ell^{-5},\ C_Z \approx 0.5$$

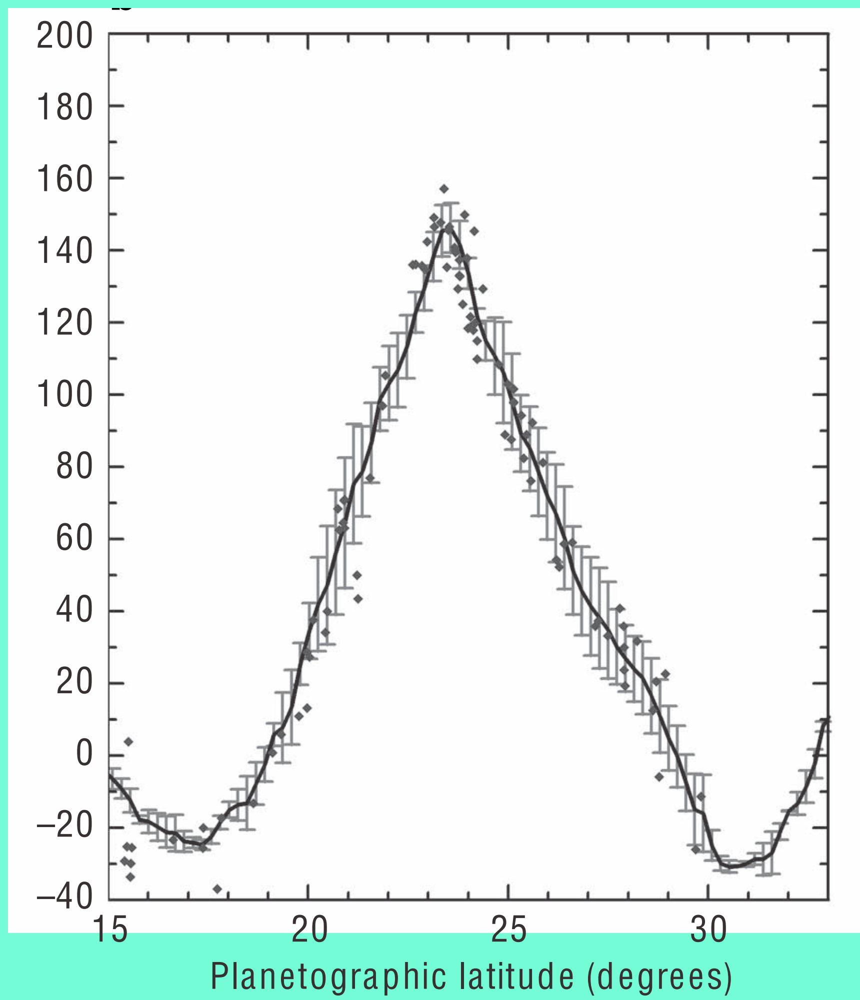


SSST equilibrium zonal flow spectral structure

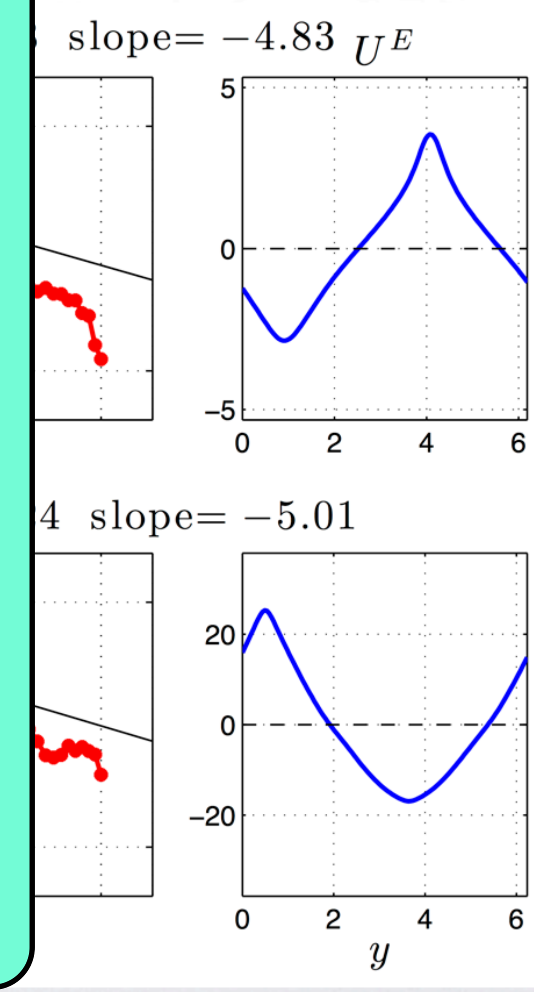
Jupiter's 23N jet

E_Z





(Sanchez-Lavega et. al., 2008)

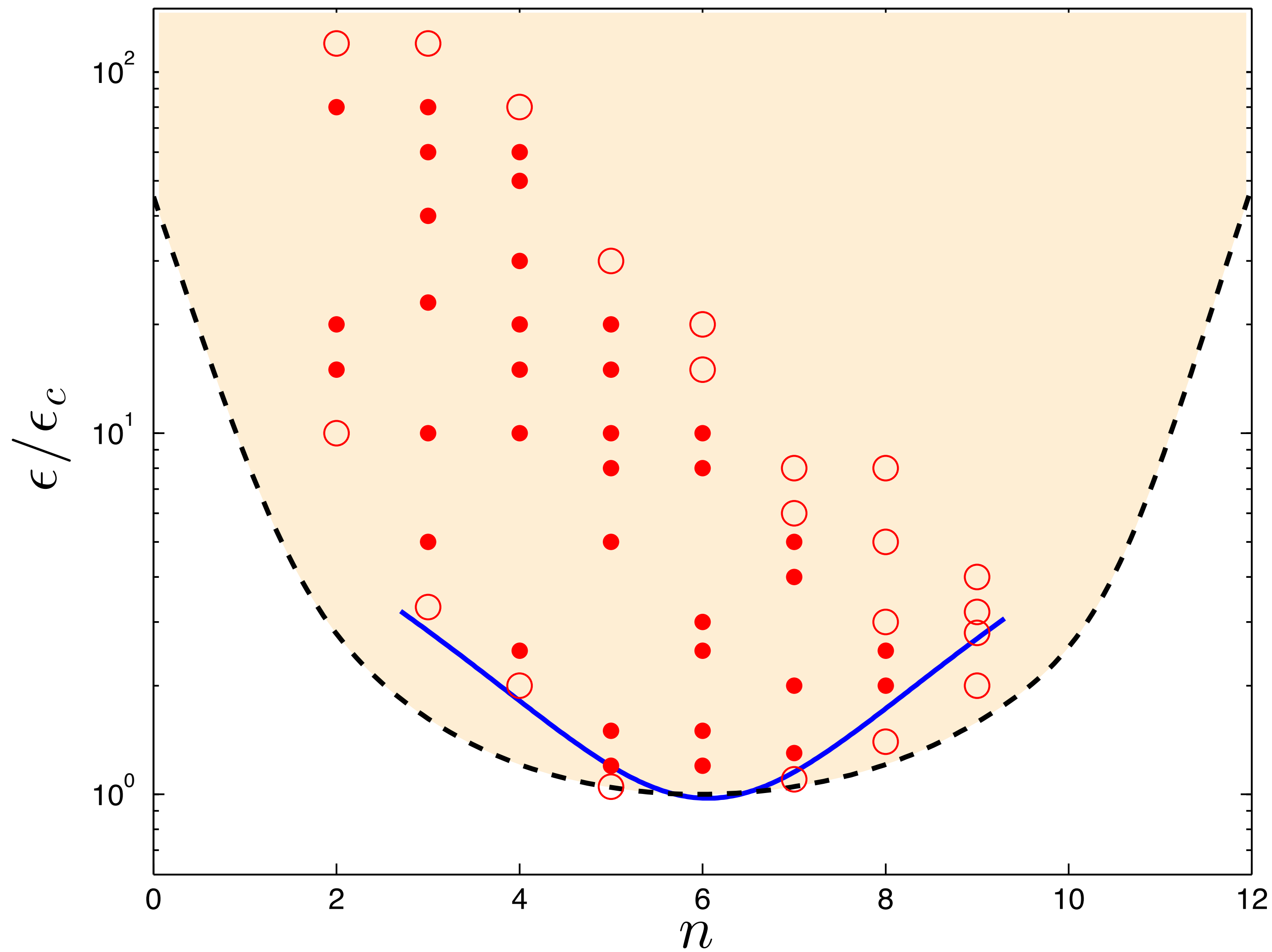


Jet merging is due to SSST instability

SSST equilibria
(stable & unstable)
are hydrodynamically stable

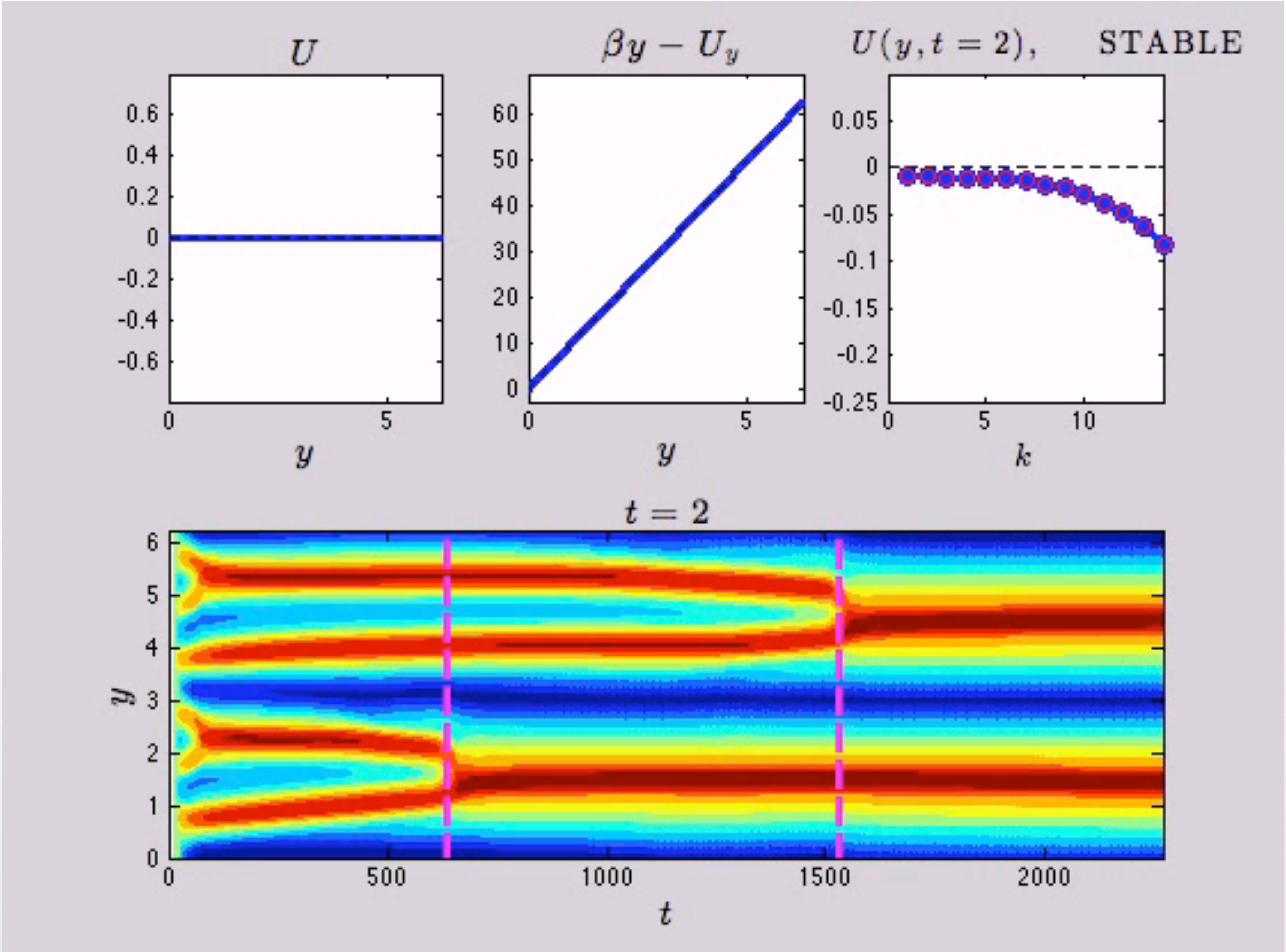
SSST instability comes prior
to hydrodynamic instability
as we move upwards in a
constant n axis

NIF at $k = 1, \ldots, 14$, $r = 10^{-1}$, $r_{\mathrm{m}} = 10^{-2}$, $\beta = 10$



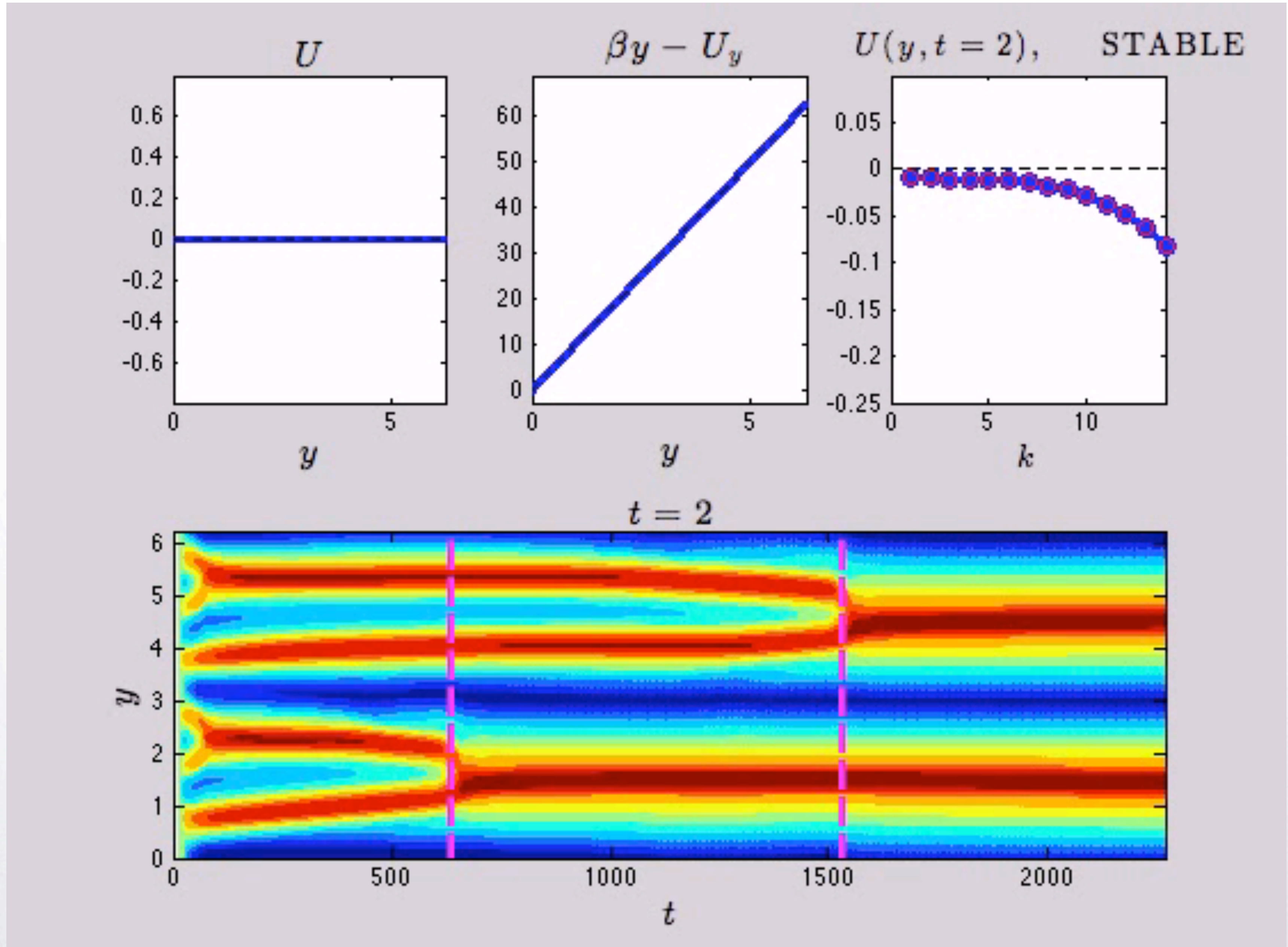
Jet merging is due to SSST instability

NIF at $k = 1, \ldots, 14$
$r = r_\mathrm{m} = 10^{-2}$, $\beta = 10$
$\epsilon = 100\epsilon_c$



Jet merging is due to SSST instability

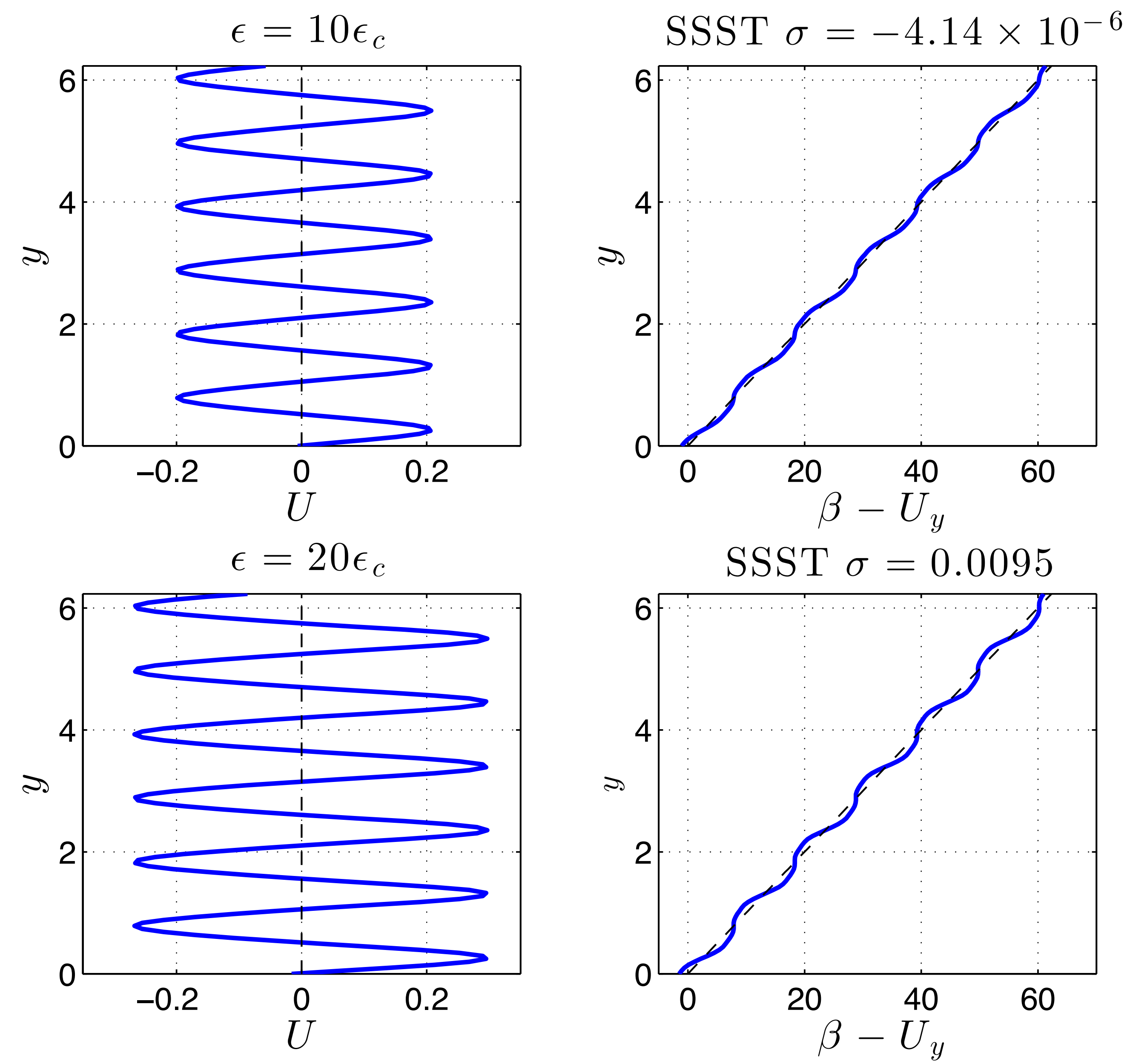
$$\text{NIF at } k = 1, \ldots, 14$$
$$r = r_{\mathrm{m}} = 10^{-2},\ \beta = 10$$
$$\epsilon = 100\epsilon_c$$



PV Staircases

NIF, $k = 1, \ldots, 14$

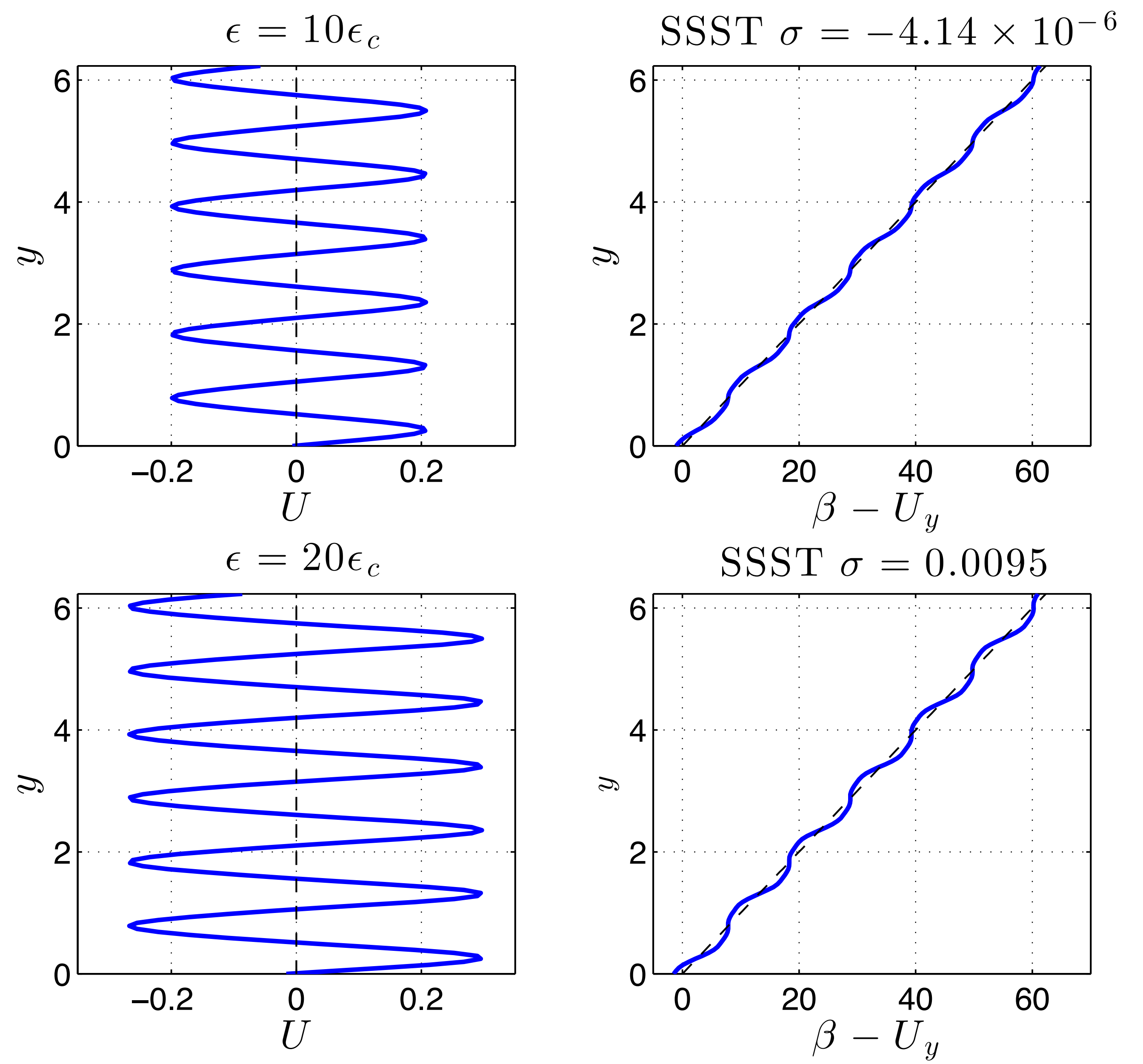
$r = 0.1$, $r_{\mathrm{m}} = r/10$, $\beta = 10$



PV Staircases

NIF, $k = 1, \ldots, 14$
$r = 0.1$, $r_{\mathrm{m}} = r/10$, $\beta = 10$

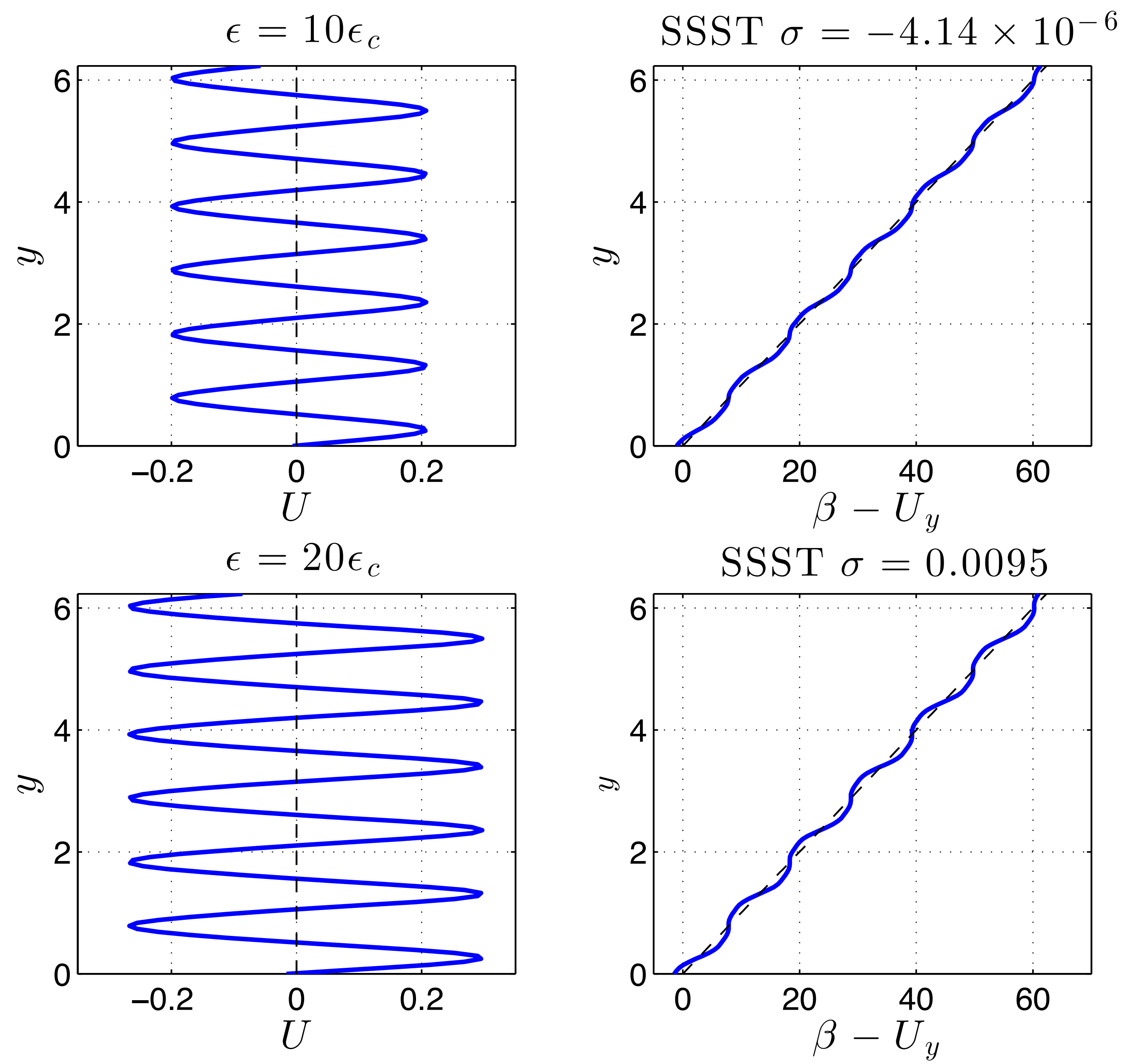
SSST stable



PV Staircases

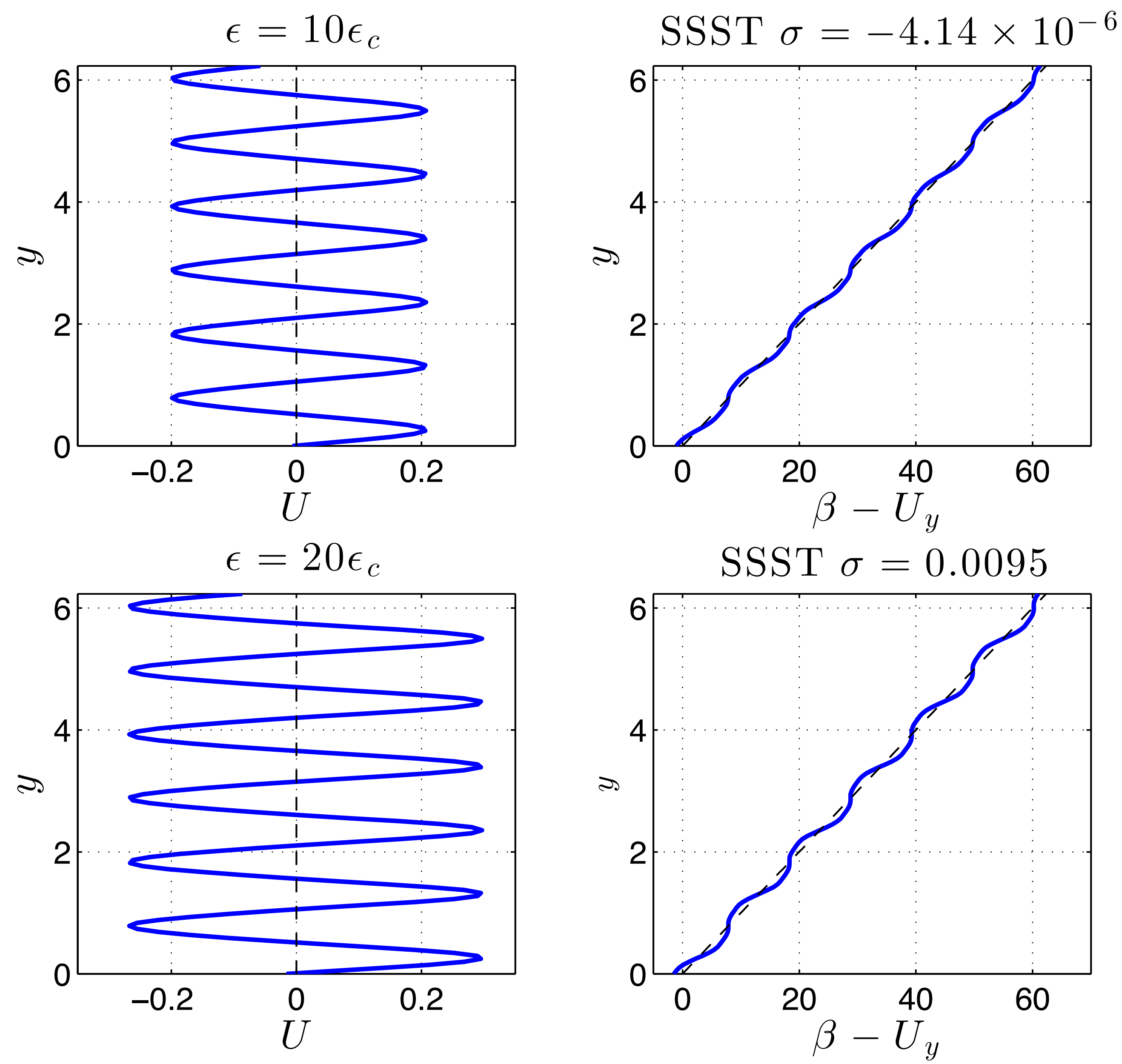
NIF, $k = 1, \ldots, 14$
$r = 0.1$, $r_\mathrm{m} = r/10$, $\beta = 10$

SSST unstable



PV Staircases

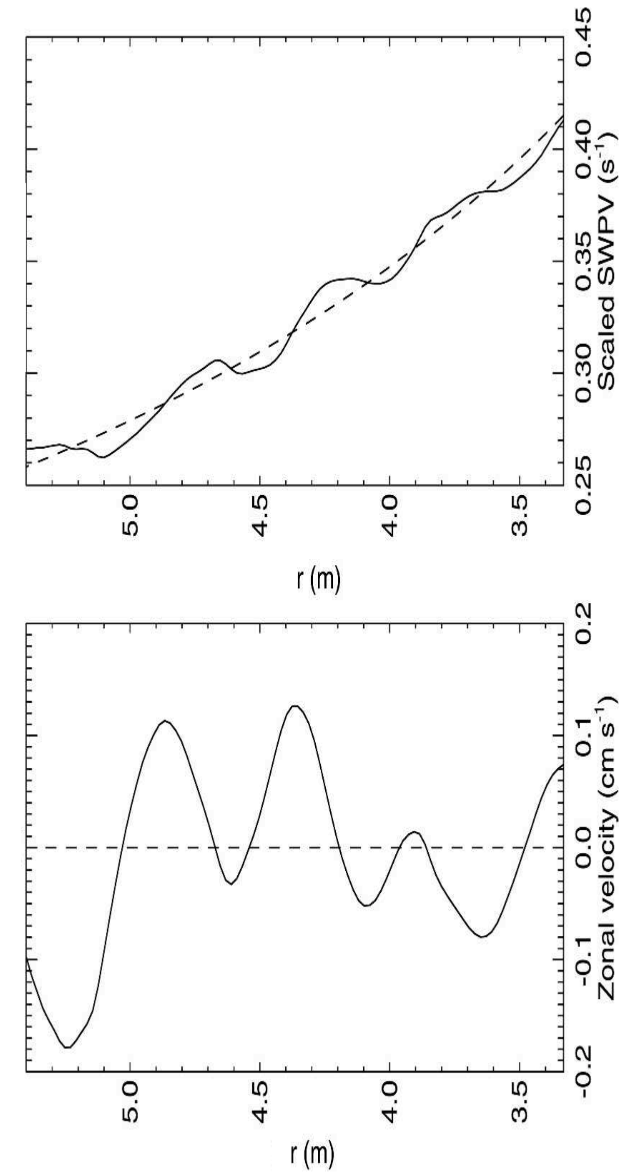
NIF, $k = 1, \ldots, 14$
$r = 0.1$, $r_{\mathrm{m}} = r/10$, $\beta = 10$



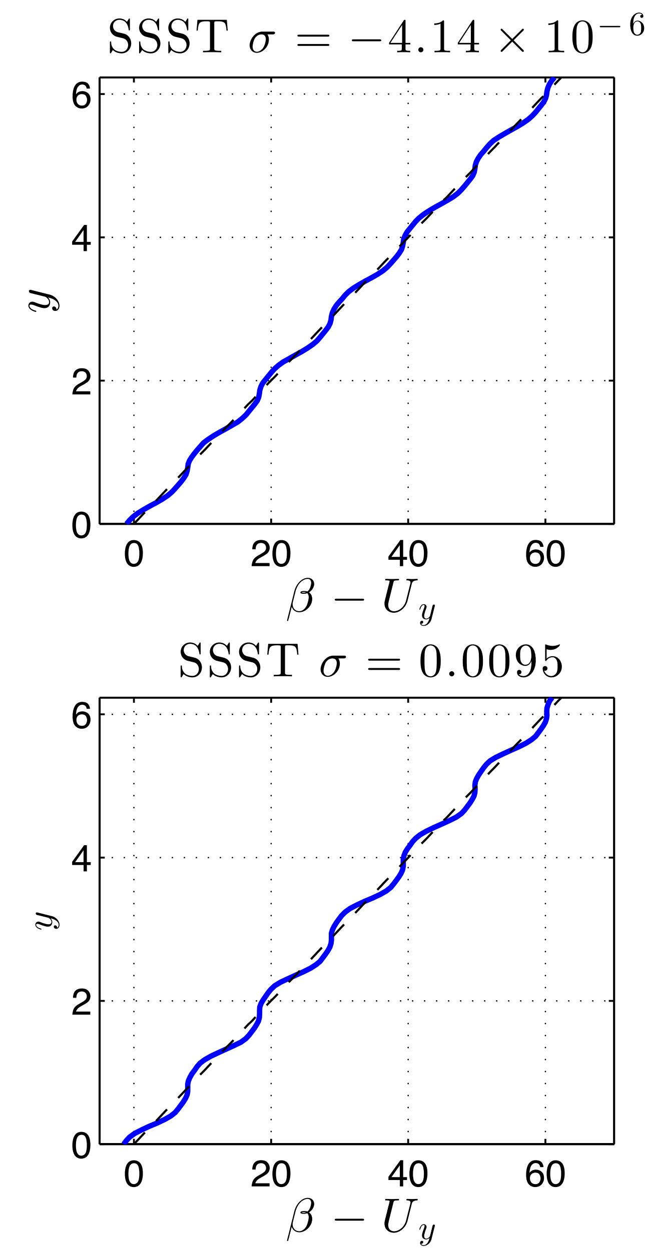
PV Staircases

NIF, $k = 1$,
$r = 0.1$, $r_{\mathrm{m}} = r/$

PV staircases in rotating tank experiment



(Read et. al., 2007)



Conclusions

Conclusions

- SSST provides prognostic theory for the emergence & equilibration of zonal flows

Conclusions

- SSST provides prognostic theory for the emergence & equilibration of zonal flows
- jet emergence in barotropic beta-plane turbulence is a result of the mean flow/perturbation SSST instability

Conclusions

- SSST provides prognostic theory for the emergence & equilibration of zonal flows
- jet emergence in barotropic beta-plane turbulence is a result of the mean flow/perturbation SSST instability
- with the modification of the turbulent spectrum due to the emergence of NZCS, SSST captures the bifurcation for jet emergence

Conclusions

- SSST provides prognostic theory for the emergence & equilibration of zonal flows
- jet emergence in barotropic beta-plane turbulence is a result of the mean flow/perturbation SSST instability
- with the modification of the turbulent spectrum due to the emergence of NZCS, SSST captures the bifurcation for jet emergence
- stochastic excitation of SSST eigenmodes reveals that these modes underlie in the dynamics even in subcritical cases

Thank you

This work has been
supported by

Onassis
Foundation

Constantinou, N.C, Ioannou, P.J. and Farrell, B.F., 2012:
Emergence and equilibration of jets in beta-plane turbulence.
(arXiv:1208.5665 [physics.flu-dyn])